NONREALIZABILITY PROOFS IN COMPUTATIONAL GEOMETRY

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Preprint-Nr. 1045 April 1987

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Abstract.

We introduce the concept of final polynomials as a systematic approach to prove nonrealizability for oriented matroids and matroids. This technique has been applied to non-polytopal spheres in previous papers [8],[12] of the first author where the resulting final polynomials were given without derivation. In the present paper the algorithmic point of view is emphasized, and we will present a method for constructing final polynomials for a certain class of oriented matroids. Our approach is motivated by the fact that final polynomials exist in every nonrealizable case [6].

Finally, we discuss a realizable simplicial 3-chirotope R_3^9 which does not admit a solvability sequence. This example shows that the coordinatizing algorithm described in [11] is not generally applicable.

1980 Mathematics Subject Classification: Primary 52A25; Secondary 05B35.

Keywords: Oriented matroid, matroid, chirotope, solvability sequence, final polynomial.

Proposed running head: Nonrealizability proofs in geometry.

* *Research of this author supported by an A.P. Sloane Doctoral Dissertation Fellowship*

1. Introduction

An interesting class of realization problems in computational geometry can be reduced to the realizability problem of chirotopes or oriented matroids, see [11]. Among such problems which have been studied in the recent literature are the polytopality of combinatorial spheres [12],[21], the embeddings of triangulated manifolds [5],[7] and the stretchability of pseudo-line arrangements [16]. On the other hand, the realizability problem for both oriented and unoriented matroids is a fundamental theoretical question in matroid theory, regardless of its applications in convexity, see e.g. [24].

So far most non-realizability proofs for matroids were based on arguments from classical projective geometry which are very specific for the particular case. The same holds for proofs that show non-polytopality for spheres.

In view of the fact that in most cases a simple characterization of the geometrically possible structures cannot be expected, see [10],[21], the algorithmic aspect becomes very important, and it seems desirable to develop *general* algebraic techniques for proving or disproving realizability of geometric objects.

It is the aim of the present paper to suggest such a technique. The method of *final polynomials* has been used already in earlier papers of the first author [8],[12], where only the results of the computations were given and applied to the geometric problem in question.

In the present paper we attempt to give a satisfactory answer to the question *how to find a final polynomial* for a given non-realizable structure. It follows from the results in [22] that deciding the realizability of arbitrary (oriented) matroids is as difficult as solving arbitrary polynomial equations and inequalities with integer coefficients, and theoretically it would be sufficient to refer to any decision procedure for real closed fields, e.g. Collins' Cylindrical Algebraic Decomposition [13]. Computer experiments of B.Kutzler show, however, that in most "real" geometric applications the corresponding inequality system is still too large for the presently implemented versions of Collins' method. On the other hand, the fact that these systems are of a very specific structure gives some hope that our techniques can be further developed along with the above methods from Computer Algebra to yield applicable decision procedures for a large class of interesting geometric problems.

We postpone the algebraic geometry underlying to our method to the last section where existence theorems for final polynomials are stated and to a subsequent joint paper of the first and third author with A.Dress [6] where these theorems will be proved in a general algebraic setting.

Throughout the paper the algorithmic point of view will be in the foreground. To explain some basic ideas we derive in the next section a final polynomial for a nonrealizable simplicial 3-chirotope with 10 points which has the structure of the Desargue-configuration. Note that the first "real" problems that have been solved by means of these methods were 4-dimensional [8],[12].

The inequality reduction to be described in Section 3 is a "geometric preprocessing" that generates a sufficiently small inequality system which still contains the entire information of the problem. Once this *reduced system* has been found, the original geometric structure can be ignored and one proceeds by variable elimination. If this elimination process leads to a contradiction, e.g. if $0 < 0$ can be derived in a certain sequence of deductive steps from the reduced system, then this sequence can be merged to a single (final) polynomial which shows the contradiction.

In Section 5 we turn to the coordinatization algorithm which has been discussed in Bokowski & Sturmfels [11]. There it has been conjectured that the absence of a solvability sequence is not sufficient for the non-realizability of a chirotope. We prove this conjecture by establishing a realizable simplicial 3-chirotope R_3^9 with 9 points which does not admit a solvability sequence. Although R_3^9 does fulfil the isotopy property [11, Sect.6], this example suggests that chirotopes with disconnected realization space might exist.

For the fundamental concepts of oriented matroids and their chirotope representation the reader is refered to [11],[14],[15].

The set of all ordered d-tuples of n elements will be denoted by

$$\Lambda(n,d) := \{(\lambda_1,\ldots,\lambda_d) \mid 1 \le \lambda_1 < \lambda_2 < \ldots < \lambda_d \le n\}.$$

Definition

A mapping $\chi : \Lambda(n,d) \to \{-1,0,+1\}$ or its unique alternating extension $\chi : \{1,\ldots,n\}^d \to \{-1,0,+1\}$ is called an (oriented) *d-chirotope with n vertices* if for all $\lambda \in \Lambda(n,d+1)$ and for all $\mu \in \Lambda(n,d-1)$ the set

$$\{(-1)^i \cdot \chi(\lambda_1,\ldots,\widehat{\lambda_i},\ldots,\lambda_{d+1}) \cdot \chi(\mu_1,\ldots,\mu_{d-1},\lambda_i) \mid i \in \{1,\ldots,d+1\}\}$$

either contains $\{-1,+1\}$ or equals $\{0\}$. The chirotope χ is called *simplicial* if $\chi(\Lambda(n,d)) \subset \{-1,+1\}$. Finally, χ is *realizable* if there exist $x_1, x_2, \ldots, x_n \in R^d$ such that

$$\text{sign}\det(x_{\lambda_1},\ldots,x_{\lambda_d}) = \chi(\lambda_1,\ldots,\lambda_d) \qquad \text{for all } \lambda \in \Lambda(n,d).$$

2. Example: A simplicial Non-Desargue chirotope

Consider the affine 3-chirotope D_3^{10} associated with the pseudo-configuration in Figure 1. It can be shown by an easy geometric argument based on Desargues theorem that D_3^{10} is not realizable, here however we want to derive a non-realizability proof for this chirotope by a final polynomial.

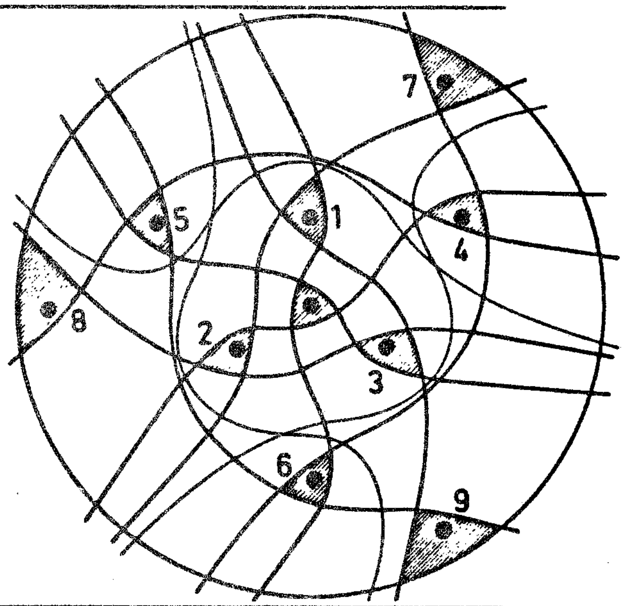

Figure 1. The non-realizable chirotope D_3^{10}.

First recall the following

Remark 2.1. (Graßmann-Plücker syzygies) *For any field* K *and* $x, y, z \in K^3$ *abbreviate the determinant* $\det(x, y, z)$ *by the bracket* $[xyz]$.

Then for all $a, b, c, d, e, f \in K^3$ we have the identities

$$\{a|bcde\} \quad := \quad [abc][ade] - [abd][ace] + [abe][acd] \quad = \quad 0$$

$$< abc|def > := [abc][abc][def] \; - \; \det \begin{pmatrix} [dbc] & [adc] & [abd] \\ [ebc] & [aec] & [abe] \\ [fbc] & [afc] & [abf] \end{pmatrix} \; = \; 0.$$

Assume that there exist points $x_1, x_2, \ldots, x_{10}$ in the real Euclidean plane such that for all i, j, k the oriented volume $[ijk]$ of the triangle x_i, x_j, x_k has the sign prescribed in Figure 1. Then the following expression vanishes by Remark 2.1.

$$
\begin{aligned}
& \{1|2345\}\,[234][153][126][126][230][137][183][293] \\
- & \{2|1340\}\,[234][153][153][126][126][137][183][293] \\
+ & \{1|2360\}\,[234][134][152][152][263][137][183][293] \\
- & \{3|1246\}\,[134][152][152][263][120][137][183][293] \\
- & \{2|1356\}\,[234][134][152][126][103][137][183][293] \\
+ & \{3|1250\}\,[234][134][152][126][126][137][183][293] \\
- & \{1|2358\}\,[134][263][137][293][120]\,\big([234][153][126]+[134][152][263]\big) \\
+ & \{2|1369\}\,[234][153][137][183][120]\,\big([234][153][126]+[134][152][263]\big) \\
- & \{3|1247\}\,[153][263][183][129][120]\,\big([234][153][126]+[134][152][263]\big) \\
+ & < 123|789 >\,[134][153][263][120]\,\big([234][153][126]+[134][152][263]\big)
\end{aligned}
$$

By expanding the Graßmann-Plücker terms $\{\ldots|\ldots\}$ and $<\ldots|\ldots>$ in the above polynomial we obtain after cancellation of several summands

$$
\begin{aligned}
& [123][145][234][153][126][126][230][137][183][293] \\
+ & [123][240][234][153][153][126][126][137][183][293] \\
+ & [123][160][234][134][152][152][263][137][183][293] \\
+ & [123][364][134][152][152][263][120][137][183][293] \\
+ & [123][256][234][134][152][126][103][137][183][293] \\
+ & [123][350][234][134][152][126][126][137][183][293] \\
+ & [123][296][234][153][137][183][120]\,\big([234][153][126]+[134][152][263]\big) \\
+ & [123][374][153][263][183][129][120]\,\big([234][153][126]+[134][152][263]\big) \\
+ & [123][185][134][263][137][293][120]\,\big([234][153][126]+[134][152][263]\big) \\
+ & \big([123]^2[789]+[172][183][293]+[172][283][139]+[182][237][139] \\
& \quad +[129][137][283]\big)\cdot[134][153][263][120]\,\big([234][153][126]+[134][152][263]\big)
\end{aligned}
$$

All brackets occuring in this polynomial, e.g. $[123],[145],\ldots,[374],[153],\ldots$, have

to be positive in any realization of D_3^{10} as is easily seen from Figure 1. Since the sum of 10 positive numbers cannot vanish, it follows that D_3^{10} is not realizable.

Such a polynomial which "obviously" shows the non-realizability of a chirotope is called a *final polynomial*, a notion that will be made precise in Section 6. In this section our main goal is to describe the steps that lead to the construction of the above final polynomial.

For that purpose we suggest to forget the above proof and assume again that D_3^{10} is realizable. Then there exists a real 3×10-matrix coordinate matrix A for D_3^{10} with

$$A = \begin{pmatrix} 1 & 0 & 0 & a & d & -g & j & -m & -p & s \\ 0 & 1 & 0 & -b & e & h & -k & n & -q & t \\ 0 & 0 & 1 & c & -f & i & -l & -o & r & u \end{pmatrix}.$$

It can be read off from Figure 1 that all variables $a, b, \ldots, u$ have to be positive. For example, we have $[124] = c > 0$ for the oriented volume $[124]$ of the triangle $1, 2, 4$. With the method in Section 3 it can be shown easily that the positivity of all 21 variables together with the following 10 inequalities forms a reduced system for D_3^{10}, i.e., D_3^{10} is realizable if and only if the inequality system (1) – (10) has a solution within the positive real numbers. We called D_3^{10} a simplicial non-Desargue chirotope because the mutations of D_3^{10}, see Section 3, have the structure of the Desargue-configuration $(10_3)_1$.

$$[123][145] = bf - ce > 0 \quad (1)$$

$$[123][185] = eo - fn > 0 \quad (2)$$

$$[123][160] = hu - it > 0 \quad (3)$$

$$[123][240] = cs - au > 0 \quad (4)$$

$$[123][256] = fg - di > 0 \quad (5)$$

$$[123][296] = ip - gr > 0 \quad (6)$$

$$[123][364] = bg - ah > 0 \quad (7)$$

$$[123][374] = ak - bj > 0 \quad (8)$$

$$[123][350] = dt - es > 0 \quad (9)$$

$$[123]^2[789] = rjn - lnp - lmq - ojp - okp - rkm > 0 \quad (10)$$

To decide the realizability of D_3^{10} we could proceed from this point on by naive variable elimination: consider for example the variable c which is contained only in (1) and in (4). All variables being positive, these two inequalities can be rewritten as

$$\frac{au}{s} < c < \frac{bf}{e}.$$

Hence a real number c satisfying (1) and (4) exists if and only if

$$bfs - aeu > 0, \quad (11)$$

and the system (1) – (10) can be replaced by the nine inequalities (2), (3), (5), (6), (7), (8), (9), (10) and (11) in one variable less.

Such elementary solving techniques can be used to finally derive $0 < 0$ which shows the contradiction. In order to obtain a final polynomial from this derivation we mimic the steps which lead to new polynomials as in (11) by forming positive linear combinations of vanishing polynomials. Therefore we consider all inequalities as (vanishing) Graßmann-Plücker identities under the additional condition that all occuring brackets are positive.

$$p_1 = [123][145] - bf + ce$$
$$p_2 = [123][185] - eo + fn$$
$$p_3 = [123][160] - hu + it$$
$$p_4 = [123][240] - cs + au$$
$$p_5 = [123][256] - fg + di$$
$$p_6 = [123][296] - ip + gr$$
$$p_7 = [123][364] - bg + ah$$
$$p_8 = [123][374] - ak + bj$$

$$p_9 = [123][350] - dt + es$$
$$p_{10} = [123][123][789] - rjn + lnp + lmq + ojp + okp + rkm$$

Solving for c as in the derivation of (11) yields the new polynomial

$$p_{11} := s\,p_1 + e\,p_4 = [123][145]s + [123][240]e - bfs + aue$$

Solving for h yields

$$p_{12} := a\,p_3 + u\,p_7 = [123][160]a + [123][364]u - bgu + ita$$

Solving for d yields

$$p_{13} := t\,p_5 + i\,p_9 = [123][256]t + [123][350]i + eis - fgt$$

Solving for t yields

$$p_{14} := fg\,p_{12} + ia\,p_{13} =$$
$$[123][160]afg + [123][364]fgu + [123][256]ait + [123][350]aii + aeiis - bfggu$$

Solving for s yields

$$p_{15} := aeii\,p_{11} + bf\,p_{14} =$$
$$[123][145]aeiis + [123][240]aeeii + [123][160]abffg + [123][364]bffgu +$$
$$[123][256]abfit + [123][350]abfii + u(aei - bfg)(aei + bfg)$$

Partially solving for j yields

$$p_{16} := rn\,p_8 + b\,p_{10} =$$
$$[123][374]rn + [123][123][789]b -$$

$$akrn + blnp + blmq + bojp + bokp + brkm$$

Partially solving for r yields

$$p_{17} := g\,p_{16} + akn\,p_6 =$$

$$[123][296]akn + [123][374]grn + [123][123][789]bg -$$

$$aiknp + bglnp + bglmq + bgjop + bgkpo + bgrkm$$

Partially solving for o yields

$$p_{18} := bgkp\,p_2 + e\,p_{17} =$$

$$[123][296]aekn + [123][374]egrn + [123][123][789]beg + [123][185]bgkp +$$

$$beg(lnp + lmq + jop + rkm) - knp(aei - bfg)$$

Finally, solving for $(aei - bfg)$ yields

$$p_{19} := u(aei + bfg)\,p_{18} + knp\,p_{15} =$$

$$[123][145]aeiiknps \;+\; [123][240]aeeiiknp \;+\; [123][160]abffgknp +$$

$$[123][364]bffgknpu \;+\; [123][256]abfiknpt \;+\; [123][350]abfiiknp +$$

$$[123][185]bgkpu(aei + bfg) \;+\; [123][296]aeknu(aei + bfg) \;+\; [123][374]$$

$$\cdot egrnu(aei + bfg) \;+ (\,[123][123][789] + lnp + lmq + jop + rkm\,)\,begu\,(aei + bfg)$$

To see that p_{19} equals the above final polynomial replace all variables by brackets according to the matrix A, e.g. a by [234], b by [134], ... etc. Under these substitutions the expressions $p_1, p_2, \ldots, p_{10}$ equal the syzygy coefficients in the first representation of the final polynomial, e.g. $p_1 = \{1|2345\}$, $p_2 = -\{2|1340\}$, which completes the argument.

3. Inequality reduction for chirotopes

We have seen in Section 2 as well as in [11],[16], that simplification to a relatively small inequality system carrying still the entire information is a very helpful first step in deciding the realizability of geometric structures. In this section we describe a method to construct a small reduced system for a given simplicial d-chirotope χ with n points. Here $\mathcal{R} \subset \Lambda(n,d)$ is called *reduced system* for χ if χ is uniquely determined by its restriction to $\mathcal{R}$, i.e., $\chi'|_{\mathcal{R}} = \chi|_{\mathcal{R}}$ implies $\chi' = \chi$ for every chirotope χ'.

We first consider d-tuples λ which are necessarily contained in every reduced system of χ. These tuples have been studied under the names *mutations* in [20] and *invertible bases* in [BdS].

As before, we use the abbreviation $\{\sigma|\tau\} := \{\sigma_1 \ldots \sigma_{d-2} \,|\, \tau_1 \ldots \tau_4\}$ for the *three term syzygies*

$$\begin{aligned} & [\sigma_1 \ldots \sigma_{d-2}\tau_1\tau_2] \cdot [\sigma_1 \ldots \sigma_{d-2}\tau_3\tau_4] \\ - & [\sigma_1 \ldots \sigma_{d-2}\tau_1\tau_3] \cdot [\sigma_1 \ldots \sigma_{d-2}\tau_2\tau_4] \\ + & [\sigma_1 \ldots \sigma_{d-2}\tau_1\tau_4] \cdot [\sigma_1 \ldots \sigma_{d-2}\tau_2\tau_3] \end{aligned}$$

where $\tau \in \Lambda(n,4), \sigma \in \Lambda(n,d-2)$. Recall that these syzygies are sufficient to define chirotopes in the simplicial case [11]. A syzygy $\{\,\sigma \,|\, \tau\,\}$ is said to *determine* a tuple $[\sigma\tau_i\tau_j]$ in χ if $\chi(\sigma,\tau_i,\tau_j)$ is uniquely determined by the values of χ for the other 5 tuples occuring in $\{\,\sigma \,|\, \tau\,\}$ and the chirotope condition for that syzygy. Given a subset $\mathcal{D} \subset \Lambda(n,d)$,
$< \mathcal{D} >$ denotes the closure of taking the union of $\mathcal{D}$ with the set of brackets $[\sigma,\tau_i,\tau_j] \in \Lambda(n,d)$ which are determined in χ by some syzygy $\{\sigma|\tau\}$ whose other 5 brackets are already determined by $\mathcal{D}$.

The idea is now to find a small subset $\mathcal{D} \subset \Lambda(n,d)$ with $<\mathcal{D}> = \Lambda(n,d)$.
$\lambda \in \Lambda(n,d)$ is a *mutation* if it is not determined by any three-term syzygy.
We write $\mathrm{Mut}(\chi)$ for the set of mutations of χ. By the results of [20] the mutations of a simplicial d-chirotope χ are in one-to-one correspondence to the sim-

plicial regions of the associated arrangement of pseudo-hyperplanes [FLa], and $|\mathrm{Mut}(\chi)| \geq n$ if χ is realizable and n is the number of points of χ.

While there are chirotopes χ for which $\mathrm{Mut}(\chi)$ is already a reduced system, e.g. the chirotope R_3^9 to be discussed in Section 5 has this property, this cannot be expected in general. The alternating chirotope $\chi_5 : \Lambda(5,3) \to \{+1\}$ has mutations $\mathrm{Mut}(\chi) := \{[123],[234],[345],[145],[125]\}$ but there is another chirotope $\chi : \Lambda(5,3) \to \{-1,+1\}$ with $\chi(1,2,3) = \chi(2,3,4) = \chi(3,4,5) = \chi(1,4,5) = \chi(1,2,5) = +1$, see Figure 2.

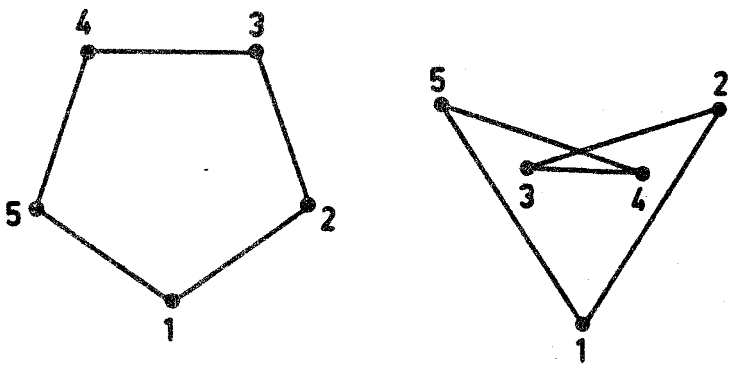

Figure 2. The chirotopes χ_5 and χ agree for all mutations of χ_5.

Since all later computations are most conveniently carried out with respect to a basis $\beta \in \Lambda(n,d)$, it is reasonable to add the set

$$V_\beta \quad := \quad \{\, [\beta_1, \ldots, \beta_{i-1}, k, \beta_{i+1}, \ldots, \beta_d] \quad | \quad k \notin \beta, i \in \{1, \ldots, d\} \,\}$$

of "variables" (with respect to β) to the set of mutations. Observe that we are free to choose among $\binom{n}{d}$ possible bases.

Remark 3.1. *Given any $\lambda \in \Lambda(n,d)$ the expansion of $[\lambda]$ with respect to a basis*

β reads

$$[\lambda] = \det \begin{pmatrix} [\lambda_1\beta_2\ldots\beta_d] & [\beta_1\lambda_1\ldots\beta_d] & \cdots & [\beta_1\beta_2\ldots\lambda_1] \\ [\lambda_2\beta_2\ldots\beta_d] & [\beta_1\lambda_2\ldots\beta_d] & \cdots & [\beta_1\beta_2\ldots\lambda_2] \\ \vdots & \vdots & \ddots & \vdots \\ [\lambda_d\beta_2\ldots\beta_d] & [\beta_1\lambda_d\ldots\beta_d] & \cdots & [\beta_1\beta_2\ldots\lambda_d] \end{pmatrix}$$

where w.l.o.g $[\beta] = 1$ is assumed.

The above $d\times d$-determinant reduces to a $k\times k$-determinant where $k = |\lambda\setminus\beta|$. To choose a suitable basis β we introduce a certain monotone "weight" function $w : N \to N$. Let $\beta \in \Lambda$ such that the expression $v(\beta) := \sum_{\lambda\in \mathrm{Mut}(\chi)} w(|\lambda\setminus\beta|)$ is minimized. In practice the following two weight functions turned out to be most useful.

- *Counting the number of "too big" determinants.*

$$w(k) = \begin{cases} 0 & \text{if } k \le m \\ 1 & \text{if } k > m \end{cases}$$

  where m is fixed.

- *Counting the total number of occurances of variables in determinants.* $w(k) := k^2$.

Now the following "filling up" algorithm determines a small reduced system.

Algorithm 3.2.

Input : Simplicial chirotope $\chi : \Lambda(n,d) \to \{-1,+1\}$, $\beta \in \Lambda(n,d)$.

Output : Small reduced system $\mathcal{R}$ for χ.

1. *Let* $\mathcal{R} := \mathrm{Mut}(\chi) \cup V_\beta$, $\mathcal{M} := \emptyset$.
2. *Determine* $\mathcal{D} := \langle \mathcal{R} \rangle$.
3. *If* $\mathcal{D} = \Lambda(n,d)$
   3.1 Then GO TO 5.
   3.2. Else : Pick $\mu \in \Lambda(n,d)\setminus\mathcal{D}$ such that $|\mu\setminus\beta|$ is minimal.
4. *Let* $\mathcal{R} := \mathcal{R} \cup \{\mu\}$, $\mathcal{M} := \mathcal{M} \cup \{\mu\}$. *Go to 2.*
5. *If* $\mathcal{M} := \emptyset$.

5.1 Then STOP, $\mathcal{R}$ is a reduced system.

5.2 Else: Pick $\lambda \in \mathcal{M}, \mathcal{M} := \mathcal{M} \setminus \{\lambda\}$

5.3 If $< \mathcal{R} \setminus \{\lambda\} > = \Lambda(n,d)$, then $\mathcal{R} = \mathcal{R} \setminus \{\lambda\}$.

5.4 GO TO 5.

There is still the freedom to choose another basis β to get a more simplified reduced system.

4. On the construction of final polynomials

Once a reduced system for an oriented matroid is constructed one is left with the problem to decide whether this system has a solution within the real numbers. In principal this decision can be made with Collins' *algebraic cylindrical decomposition* method, see [13], which is currently the best known decision procedure for real algebraic varieties.

In praxis, however, there are still long ways to go before Collins' method can be applied to solve "real" problems. Consider the case of a symmetric embedding of Möbius' torus with 7 vertices in dimension 3 see [7]. This simplest of all examples we were interested in could be reduced to the following system in four variables.

$$d > 0, \quad b - 1 > 0, \quad -c > 0, \quad a - b + ac > 0,$$

$$b - ad > 0, \quad ad + bc - bd > 0, \quad b + c + d - ad - 1 > 0$$

B. Kutzler solved this system with an implementation of Collins' method at the University of Linz in around 2 hours of CPU time, and slightly bigger systems could not be solved within 24 CPU hours. Still there is some hope that the algorithm can be significantly improved for our special type of problems, and we expect better results in the near future.

In this section we describe the underlying idea which lead so far to a collection of polynomials in the syzygy ideal which finally tell us that the oriented matroids under consideration are not realizable.

The existence of such a *final polynomial* in case of nonrealizability is a consequence of a real version of Hilbert's Nullstellensatz, see [6].

But to find the final polynomial is difficult and our approach is so far in general not completely successful from a practical point of view.

So far final polynomials were applied in different geometrical problems of interest such as the smallest nonrealizable matroid polytope M^9_{963}, [1], [2], in case of Altshuler's sphere M^{10}_{425}, [8], in the two examples given in this paper, and in case of a search for symmetrical realizations of a manifold with minimal number of vertices [B87].

In all these cases the reduced system was small enough, and the solving technique used to find a solvability sequence or used in the above example to get inequality (11) etc. succeeded and was afterwards translated as above into a suitable polynomial in the syzygy ideal to get a final polynomial.

Perhaps it was the special structure of these examples, homogeneous in nature, integer coefficients and linear occurance of all variables at the beginning together with a careful check how to start to solve for variables which lead to a decision by hand in all these cases.

Computer-aided implementations using these properties or/and a special variant of Collins' method might bring a substantial improvement in this direction.

5. A configuration without solvability sequence

In [11] the first and the third author introduced the concept of solvability sequences as an algorithmically oriented sufficient criterion for realizability of simplicial chirotopes. This criterion has been used to decide geometric realizability in a large collection of instances [12]. It is an important property of chirotopes with solvability sequence that their realization space is contractible [11, Theorem 5].

Although an affirmative answer to that question was extremely unlikely, it remained an open problem whether all realizable chirotopes do have a solvability sequence, in which case nonrealizability of simplicial chirotopes could have been proven by exhaustive search over all variable orderings. In this section we disprove this conjecture by giving an example due to the second author of a simplicial 3-chirotope with 9 vertices which does not admit a solvability sequence.

Recall the following definitions from [11, Section 5]. Let χ be a simplicial d-chirotope with n elements. Pick a *basis* $\beta \in \Lambda(n,d)$, and let, as in Section 3, V_β denote the set of (bracket) variables with respect to this basis. Viewing the $\Delta \in \Lambda(n,d)$ as coordinate functions on the realization space $\mathcal{O}_\chi$ of χ, every $\Delta \in \Lambda(n,d)$ can be expressed as a determinant in certain variables from V_β by Remark 3.1.

For a given total order $(v_1, \ldots, v_{d(n-d)})$ of the set of variables V_β define

$$\Lambda_i \quad := \quad \left\{ \Delta \in \Lambda(n,d) \;:\; \frac{\partial \Delta}{\partial v_j} = 0 \quad \text{for } j > i \right\}$$

The sequence $(v_1, \ldots, v_{d(n-d)})$ is called a *solvability sequence* for χ if the following condition holds.

For all $\eta_1, \ldots, \eta_{i-1} \in R$ with

$$\operatorname{sign} \Delta(\eta_1, \ldots, \eta_{i-1}) \quad = \quad \chi(\Delta) \qquad \text{for all } \Delta \in \Lambda_{i-1}$$

there exists an $\eta_i \in R$ such that

$$\operatorname{sign} \Delta(\eta_1, \ldots, \eta_i) \quad = \quad \chi(\Delta) \qquad \text{for all } \Delta \in \Lambda_i$$

In other words : $(v_1, \ldots, v_{d(n-d)})$ is a solvability sequence for χ if the "greedy" type assignment of real numbers η_i to the variables v_i in the prescribed order necessarily yields a coordinatization of χ.

Naturally, a similar definition makes sense on the level of points. We call a d-chirotope χ on an n-element set E *reducible* if either $|E| = d$, that is, χ is a simplex, or there is a point $e \in E$ such that for all $X \subset R^d$ with $X \in \mathcal{O}_{\chi\setminus\{e\}}$ there exists an $x \in R^d$ such that $X \cup x \in \mathcal{O}_\chi$. Notice that every 2-chirotope is reducible. The motivation for this definition is that it suffices to consider only nonreducable cases in order to decide realizability.

Consider the 3-chirotope R_3^9 associated with the point configuration in Figure 3.

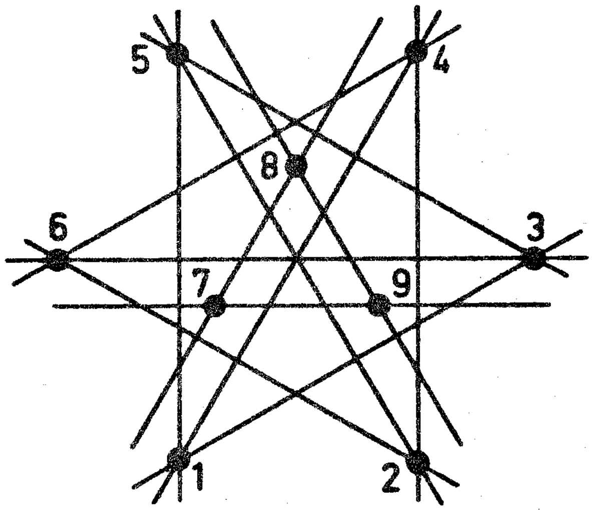

Figure 3. The simplicial chirotope R_3^9 without solvability sequence.

Theorem 5.1. *The realizable chirotope R_3^9 is not reducible.*

Proof. Since R_3^9 has the combinatorial symmetry $\sigma = (123456)(789)$ it is sufficient to consider representatives from the two orbits of σ, say, point 1 and point 8. In Figure 4a,b realizations of $R_3^9 \setminus 1$ and $R_3^9 \setminus 8$ are given, which cannot be extended to a realization of R_3^9. To see this, consider the darkened lines in Figure 4a, 4b.

It is easy to check that there is no point satisfying all orientation conditions as indicated by the arrows at the darkened lines simultaneously. Hence by definition R_3^9 is not reducible.

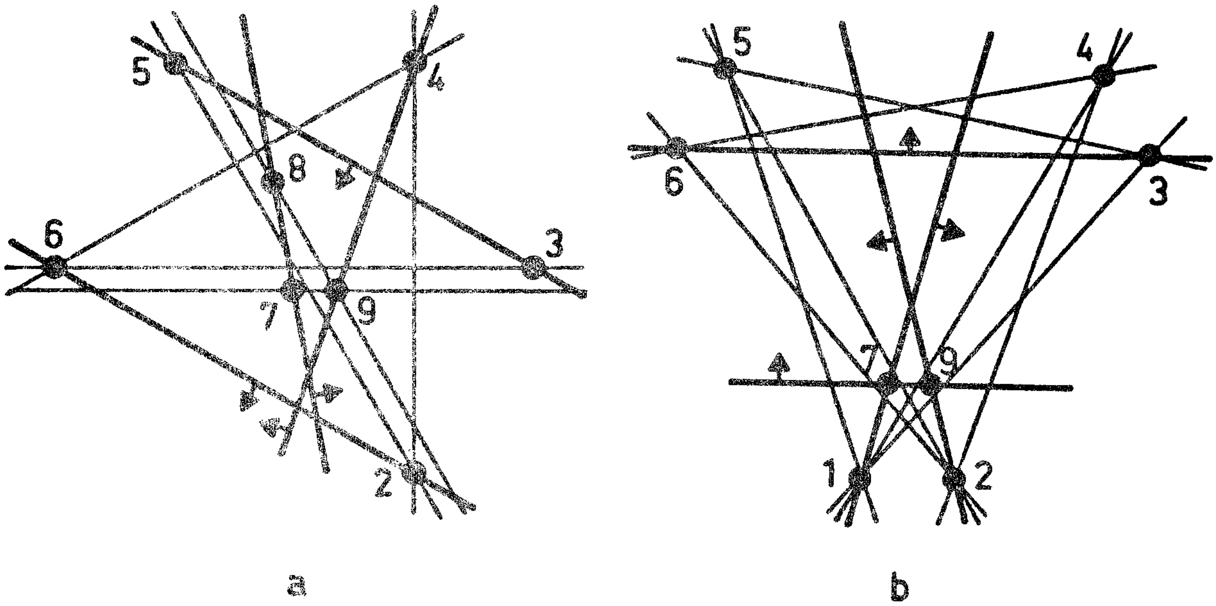

Figure 4. R_3^9 is not reducible.

Notice that the line arrangement associated with the oriented matroid R_3^9 can be obtained by deletion from the unique tight arrangement with 10 lines, see [19, Figure 5], which also is not reducible. An oriented matroid χ being *tight* means that every mutation of a rank 2 minor of χ induces a mutation of χ.

Proposition 5.2. *If a simplicial 3-chirotope χ admits a solvability sequence then χ is reducible.*

Proof. Let χ be a simplicial 3-chirotope with n points, and assume that for some basis $\beta \in \Lambda(n,3)$ the set of variables V_β can be ordered to give a solvability sequence for χ. Suppose that $v_{3(n-3)} := (\beta \setminus \beta_i) \cup k\,,\; k \notin \beta$, is the last element of this solvability sequence. As above, we write

$$\Lambda_{3(n-3)-1} \quad := \quad \{\, \Delta \in \Lambda(n,d) \,:\, \frac{\partial \Delta}{\partial v_{3(n-3)}} = 0 \,\}.$$

To see that χ is reducible, we need to show that every realization $X \subset R^3$ of $\chi \setminus k$ extends to a realization $X \cup x_k \subset R^3$ of χ. Every rank 2 oriented matroid being reducible, the induced realization of $(\chi/\beta_i)\setminus k$ can be extended to a realization of (χ/β_i). This means that we can assign real numbers to the variables $[(\beta \setminus \beta_j) \cup k]$, $j \in \{1,2,3\}\setminus\{i\}$ in a way that is compatible with χ. This assignment together with X gives us real numbers $\eta_1, \ldots, \eta_{3(n-3)-1}$ such that $\text{sign}(\Delta) = \chi(\Delta)$ for all $\Delta \in \Lambda_{3(n-3)-1}$.

Since $v_{3(n-3)}$ was assumed to be the last element in a solvability sequence, we can find an $\eta_{3(n-3)} \in R$ which completes the sequence $\eta_1, \ldots, \eta_{3(n-3)-1}$ to a realization of χ. Retranslated in geometric language: every realization of $\chi \setminus k$ extends to a realization of χ. This proves the claim.

Theorem 5.1 together with Proposition 5.2 implies

Corollary 5.3. *The realizable chirotope R_3^9 does not admit a solvability sequence.*

Let us close this section with the remark that although R_3^9 does not admit a solvability sequence, it fulfills the isotopy property. This will be shown in another paper. Without proof of correctness we give a short nondeterministic construction algorithm for all realizations of R_3^9. This construction procedure shows furthermore that the realization space of R_3^9 is contractible, hence path connected.

1. Realize $R_3^9 \setminus \{1,6\}$. (Every 3-chirotop with 7 points is realizable).
2. Delete point 7.
3. Insert points 1 and 6.
4. Insert point 7.

In Figure 5a,b,c an example of this construction sequence is given.

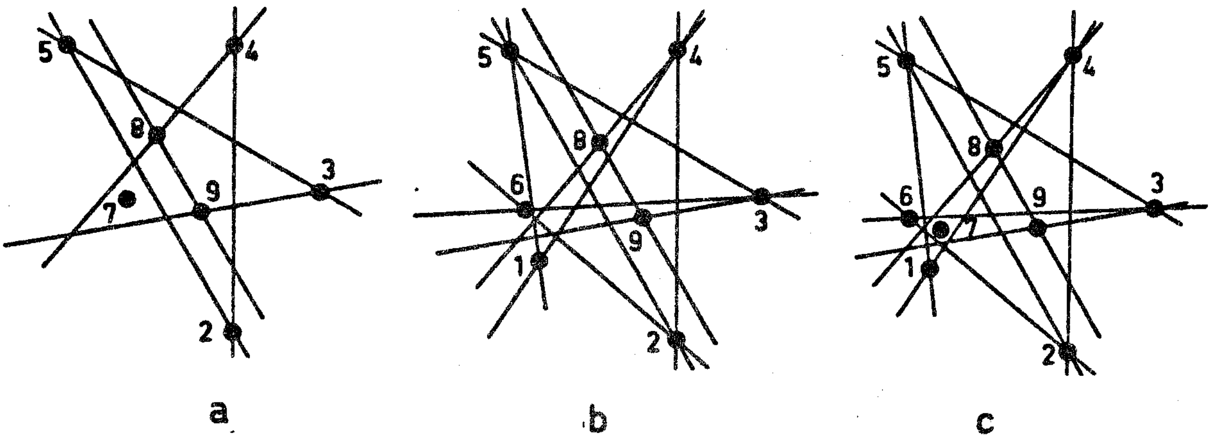

Figure 5. Geometric construction showing the isotopy property for R_3^9.

6. Existence of final polynomials for matroids and oriented matroids

The main subject of this paper is the construction of final polynomials as non-realizability proofs for oriented matroids. Naturally, it is a fundamental question whether this method is generally applicable, i.e., whether for every non-realizable object there exists a final polynomial.

Using methods from real algebraic geometry it can be shown that the answer to this question is "yes". A discussion of the algebraic details and proofs for all theorems in this section will be contained in a forthcoming paper by Bokowski, Dress and Sturmfels [6]. Here we restrict ourselves to give the precise definition of *final polynomials* for matroids and oriented matroids and to state without proof the existence results for final polynomials. We also include a final polynomial proof for Pappus' theorem as an example for the unoriented case.

Given integers $n \geq d \geq 1$ and a field K, consider the ring $K[\Lambda(n,d)]$ freely generated over K by all brackets $[\lambda]$, $\lambda \in \Lambda(n,d)$.

As usual, we write $[\lambda_{\pi(1)} \ldots \lambda_{\pi(d)}] := \operatorname{sign} \pi \cdot [\lambda_1 \ldots \lambda_d]$ for any permutation π.

Let $I^K_{n,d}$ denote the ideal generated in $K[\Lambda(n,d)]$ by all quadratic syzygies

$$\sum_{i=1}^{d+1}(-1)^i \cdot [\lambda_1,\ldots,\lambda_{i-1},\lambda_{i+1},\ldots,\lambda_{d+1}] \cdot [\lambda_i,\mu_1,\ldots,\mu_{d-1}],$$

where $\lambda \in \Lambda(n,d+1), \mu \in \Lambda(n,d-1)$.
Viewing $K[\Lambda(n,d)]$ as the ring of polynomial functions on the $\binom{n}{d}$–dimensional vector space $\wedge_d K^n$, $I^K_{n,d}$ is the vanishing ideal of the Graßmann variety of simple d-vectors. The coordinate ring $K[\Lambda(n,d)]/I^K_{n,d}$ of this variety is, in the terminology of N.White [23], the *bracket ring* with coefficients in K of the uniform rank d matroid on an n-element set.

Now let M be any rank d matroid on $E = \{1,2,\ldots,n\}$. We assign to M the two sets of all bracket polynomials that must (resp. cannot) vanish under a coordinatization. Let I^K_M denote the ideal in $K[\Lambda(n,d)]$ which is generated by
$\{ [\lambda], \lambda$ dependent in $M \}$,
and let S^K_M denote the multiplicative semigroup with unit generated by
$\{ [\lambda], \lambda$ is basis of $M \}$.
In other words, I^K_M consists of all linear combinations of non-basis brackets with polynomial coefficients, and S^K_M consists of all non-zero monomials which are products of basis brackets.

A polynomial $f \in K[\Lambda(n,d)]$ is called a *final polynomial* for M if $f \in I^K_{n,d} \cap (S^K_M + I^K_M)$. With this definition we can state the desired "theorem of the alternate" for realizabilty of matroids.

Theorem 6.1. *Let M be a matroid and K a field.*

Then one and only one of the following statements is true.
(i) There exists a final polynomial for M with coefficients in K.
(ii) M is realizable over some finite algebraic field extension of K.

In particular, we have

Corollary 6.2. *A matroid M is not realizable over an algebraically closed field K if and only if there exists a final polynomial for M with coefficients in K.*

Example 6.3. *The non-Pappus matroid NP, see Figure 6, is the rank 3 matroid on $E = \{1,2,\ldots,9\}$ defined by its non-bases (three point lines)*

$$\mathcal{N} \quad = \quad \{ \ [129],[138],[156],[345],[489],[579],[678],[237] \ \}.$$

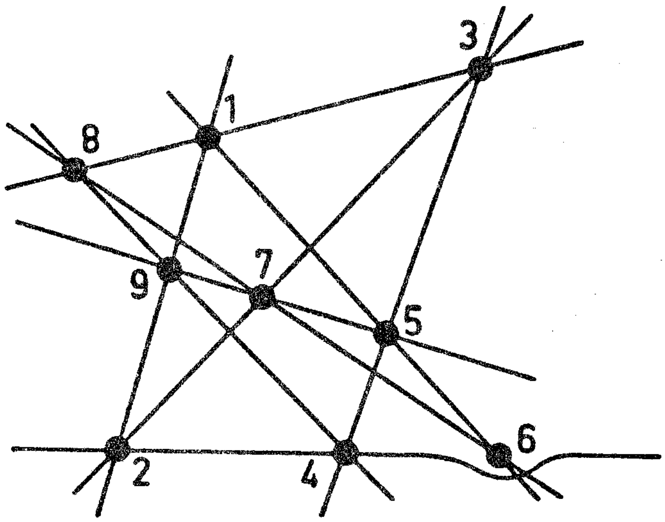

Figure 6. The non-Pappus matroid NP.

This matroid is not realizable over any field K. For, assume there exist $x_i \in K^3, i = 1,\ldots,9$ such that x_i, x_j, x_k are linearly dependent if and only if $[ijk] \in \mathcal{N}$. Then x_1, x_3, x_8 and x_5, x_7, x_9 being both dependent implies by Pappus' theorem that the three vectors $x_2 = (x_1 \vee x_9) \wedge (x_3 \vee x_7)$, $x_4 = (x_3 \vee x_5) \wedge (x_8 \vee x_9)$ and $x_6 = (x_1 \vee x_5) \wedge (x_8 \vee x_7)$ are linearly dependent as well; in contradiction to $[246] \notin \mathcal{N}$.

Let us give a non-realizability proof for NP (and hence a proof for Pappus' theorem) by establishing a final polynomial. The polynomial

$$\begin{aligned} p \;:=\; & \{4|1267\}\,[148][157][437][197] + \{1|2479\}\,[148][157][437][467] \\ & + \{1|3478\}\,[149][247][157][467] + \{1|4567\}\,[148][247][347][197] \\ & + \{4|1357\}\,[148][427][167][179] + \{4|1789\}\,[247][157][167][413] \\ & + \{7|1459\}\,[247][148][167][143] + \{7|1468\}\,[247][157][149][134] \\ & + \{7|1234\}\,[194][148][157][467] \end{aligned}$$

is a linear combination of three term syzygies and hence contained in the above defined ideal $I^K_{9,3}$. By expanding the expressions $\{i|jklm\}$ we obtain a sum of 27 monomials of degree six, 18 of which vanish by pairwise cancellation. It remains

$$\begin{aligned} p \;=\; & [246][147][148][157][437][197] + \underline{[129]}[147][148][157][437][467] \\ & + \underline{[138]}[147][149][247][157][467] + \underline{[156]}[147][148][247][347][197] \\ & - \underline{[435]}[147][148][427][167][179] - \underline{[489]}[147][247][157][167][413] \\ & + \underline{[759]}[147][247][148][167][143] + \underline{[768]}[147][247][157][149][134] \\ & + \underline{[723]}[147][194][148][157][467] \end{aligned}$$

Since all underlined brackets are contained in $\mathcal{N}$, the last eight summands are contained in the ideal I^K_{NP}. On the other hand, $[246][147][148][157][437][197] \in S^K_{NP}$, and hence $p \in I^K_{9,3} \cap (S^K_{NP} + I^K_{NP})$, that is, p is a final polynomial for the Non-Pappus matroid NP.

Considering the above bracket ring and ideals with integer coefficients we obtain the following generalization of this example.

Remark 6.3. *A matroid M is not realizable over any K if and only if there exists a final polynomial for M with integer coefficients.*

Let us now turn to the case of oriented matroids or chirotopes. Therefore assume that K is an ordered field, and let $K[\Lambda(n,d)]$ and $I^K_{n,d}$ as above. Given a (not necessarily simplicial) d-chirotope χ on $E = \{1, 2, \ldots, n\}$ we assign to χ the three sets I_χ^K, N_χ^K and P_χ^K of bracket polynomials which are respectively zero, nonnegative and positive in any realization of χ. First let (as above) I_χ^K denote the ideal in $K[\Lambda(n,d)]$ which is generated by $\{\, [\lambda] \in \Lambda(n,d)\,,\, \chi(\lambda) = 0 \,\}$. For simplicial chirotopes we have clearly $I_\chi^K = 0$.

Let N_χ^K denote the multiplicative semigroup with unit generated the positive brackets $\{\, [\lambda]\,,\, \chi(\lambda) = +1 \,\}$, the negated negative brackets $\{\, -[\lambda]\,,\, \chi(\lambda) = -1 \,\}$ and the positive elements in the ordered field K. Finally, define P_χ^K to be the quadratic semiring in $K[\Lambda(n,d)]$ which is generated by N_χ^K and the set $K[\Lambda(n,d)]^2$ of all squares.

A polynomial $f \in K[\Lambda(n,d)]$ is called a *final polynomial* for χ if $f \in I_{n,d}^K \cap (I_\chi^K + N_\chi^K + P_\chi^K)$. (Compare with the example in Section 2.) With this definition we have

Theorem 6.4. *Let χ be a chirotope and K an ordered field. Then one and only one of the following statements is true.*

(i) There exists a final polynomial for M with coefficients in K.

(ii) M is realizable over some finite algebraic ordered field extension of K.

Since every realizable chirotope (over some ordered field) is realizable over the real algebraic numbers we have

Corollary 6.5. *A chirotope χ is not real realizable if and only if there exists a final polynomial for χ with rational coefficients.*

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