EXTREMAL PROPERTIES OF 0/1-POLYTOPES

ULRICH HUND, JÜRGEN RICHTER-GEBERT*, A. SARANGARAJAN,
AND GÜNTER M. ZIEGLER*

Abstract. We provide lower and upper bounds for the maximal number of facets of an d-dimensional 0/1-polytope, and for the maximal number of vertices that can appear in a 2-dimensional projection ("shadow") of such a polytope.

1. Introduction

The combinatorics of 0/1-polytopes is at the core of many investigations of Combinatorial Optimization. In fact the field of "Polyhedral Combinatorics" is concerned with classes of facets and other combinatorial structure of "special" 0/1-polytopes that are given as the convex hulls of the characteristic vectors of solutions of certain problem classes. In particular — just to mention one well-studied classical case — quite a lot is known about the facet structures of traveling salesman polytopes: see Grötschel & Padberg [6].

Much less is known about "general" 0/1-polytopes. However, it seems that the "special" polytopes of Combinatorial Optimization can't be much simpler: so Billera & Sarangarajan [2] have recently demonstrated that in the very special class of asymmetric traveling salesman polytopes one might encounter every 0/1-polytope as a face.

In the following, we discuss two classes of extremal problems for general 0/1-polytopes that arise from complexity considerations of Combinatorial Optimization.

1.1. **The maximal number of facets.** The first section of the Grötschel and Padberg [6] chapter on "Polyhedral Computations" for the traveling salesman problem is titled "1.1: The number of facets of TSP polytopes and algorithmic implications." Grötschel & Padberg note that traveling salesman polytopes have "many" facets. To get a better notion of "many," estimates on the numbers of facets of general 0/1-polytopes are needed. Grötschel & Padberg use a very crude upper bound, namely that a d-dimensional 0/1-polytope cannot have more than

$$f(d) \quad \leq \quad \binom{2^d}{d} \approx 2^{d^2}$$

facets, since every facet is determined by a set of d vertices. A much better bound was given by Bárány [13, Problem 0.15*]: $f(d) \leq d! + 2d$. Below — in Section 2 — we slightly improve Bárány's bound, to

$$f(d) \quad \leq \quad (d-1)((d-1)! + 2)$$

for $d \geq 3$.

Still, all the lower bounds we can offer are singly exponential. While $f(d) \geq 2^d$ is easy to see (from the cross polytopes realized as 0/1-polytopes), we obtain

$$f(d) \quad \geq \quad (2.5)^d$$

for all sufficiently high d.

* Supported by a DFG Gerhard-Hess-Forschungsförderungspreis.

So, what does "many facets" mean? Let's take the (symmetric) traveling salesman polytopes Q_T^n as our benchmark, a polytope of dimension $d = n(n-3)/2$ with $(n-1)!/2$ vertices. For $n = 8$ this is a 20-dimensional polytope with $194,187 \approx (1.8383)^{20}$ facets [4], while we can construct a polytope S_{20} of dimension 20 with as many as

$$f(20) \geq 117,267,241 \approx 10^8 > (2.5)^{20}$$

facets. Still, the upper bound we have gives

$$f(20) \leq 1216451004088320020 \approx 10^{18}.$$

Similarly, in the case of 120 city problems, the TSP-polytope Q_T^n has dimension $d = 7020$. The number of facets of this polytope is not known; Grötschel & Padberg note that a class more than $2{\cdot}10^{179} \approx (1.0606)^d$ facets ("comb constraints") is known. At the same time, we can construct a 0/1-polytope S_{7020} of the same dimension that have as many as $10^{2836} \approx (2.53)^d$ facets.

1.2. **The size of a 2-dimensional shadow.** For any class of polytopes $\mathcal{P}$ one has the following quantities:

$M(\mathcal{P})$: the maximal number of vertices,

$H(\mathcal{P})$: the maximal number of vertices on a path that is strictly increasing with respect to a linear function (an *increasing* path),

$H_{sh}(\mathcal{P})$: the maximal number of vertices on a 2-dimensional projection ("shadow").

For the class $\mathcal{P}_d$ of all d-dimensional 0/1-polytopes we have

$$\frac{1}{2}(H_{sh}(\mathcal{P}_d) - 1) \quad \leq \quad H(\mathcal{P}_d) \quad \leq \quad M(\mathcal{P}_d) = 2^d.$$

For the class $\mathcal{P}(d,n)$ of d-dimensional polytopes with at most n facets this hierarchy has been analyzed in [1].

In Section 3 we give exponential (lower and upper) bounds for the quantity $H_{sh}(\mathcal{P}_d)$. The motivation for this comes from Linear Programming. In fact, $\lceil \frac{1}{2} H_{sh}(PP_d) \rceil$ is the maximal number of non-degenerate pivots that the simplex algorithm, with the shadow boundary (or Gass-Saaty) pivot rule [3], can take on a 0/1 problem. This is 1 less than the maximal number of different basic solutions (i.e., vertices of the polytope) that the algorithm may visit. (However, since 0/1-polytopes are in general very degenerate, this does not bound the maximal number of pivots, or of basic solutions encountered.)

Is there any polynomial augmentation method on 0/1-polytopes? This is of interest since edge paths of polynomial length can be constructed from any augmentation oracles (as in [10]) that would output only augmentation vectors that correspond to edges. Is there *any* strategy that on a 0/1-polytope would need only a polynomial number of nondegenerate pivots?

2. The maximal number of facets

Let $f(d)$ be the largest number of facets of a d-dimensional 0/1-polytope. (It is easy to see that for this we need only consider d-dimensional 0/1-polytopes $P \subseteq \mathbb{R}^d$.) Also, let $f'(d)$ denote the same number under the additional assumption that P contains the center $(\frac{1}{2})^d$ of the d-dimensional 0/1-cube in its interior; we have $f'(d) \leq f(d)$ for all d, by definition. For small dimensions we have the following values (derived below):

$$\begin{array}{rcccccl} & & f'(d) & = & f(d) & = & 2^d \quad \text{for } d \leq 4, \\ 40 & \leq & f'(5) & \leq & f(5) & \leq & 104, \\ 121 & \leq & f'(6) & \leq & f(6) & \leq & 610. \end{array}$$

We use the following "direct sum" construction.

Proposition 2.1. *For $i = 1, 2$ let $P_i = \text{conv}(V_i) \subseteq \mathbb{R}^{d_i}$ be d_i-dimensional 0/1-polytopes. that have the center $(\frac{1}{2})^{d_i}$ of the d_i-dimensional 0/1-cube in the interior. Then there is a $(d_1 + d_2)$-dimensional 0/1-polytope, denoted*

$$P_1 \oplus P_2 := \text{conv}(V_1) \oplus \text{conv}(V_2) \subseteq \mathbb{R}^{d_1+d_2},$$

called the direct sum *of V_1 and V_2, that has $(\frac{1}{2})^{d_1+d_2}$ in its interior, and that has*

$$f_{d_1+d_2-1}(P_1 \oplus P_2) = f_{d_1-1}(P_1) \cdot f_{d_2-1}(P_2)$$

facets.

Proof. We use the embedded 0/1-cubes

$$\begin{array}{llll} \text{conv}\{x \in \{0,1\}^{d_1+d_2} & : x_1 = x_2 = \ldots = x_{d_1} = x_{d_1+1}\} & =: C'_{d_2} & \cong C_{d_2} \\ \text{conv}\{x \in \{0,1\}^{d_1+d_2} & 1 - x_{d_1} = x_{d_1+1} + \ldots = x_{d_1+d_2}\} & =: C'_{d_1} & \cong C_{d_1} \end{array}$$

in the (d_1+d_2)-dimensional 0/1-cube that are positioned in two orthogonal affine subspaces of $\mathbb{R}^{d_1+d_2}$ that intersect in $(\frac{1}{2})^{d_1+d_2}$. Lifting P_1 and P_2 to subpolytopes of C'_{d_1} resp. C'_{d_2} we obtain the usual "free sum" construction for polytopes (cf. [8] [7]) as a construction for "centered 0/1-polytopes." □

Starting from $P_1 = [0,1] \subseteq \mathbb{R}$ and $f'(1) = f(1) = 2$ we thus obtain a d-dimensional 0/1-polytope

$$C_d^{\Delta'} := P_1 \oplus P_1 \oplus \ldots \oplus P_1$$

with 2^d facets that realizes the d-dimensional cross polytope as a 0/1-polytope:

$$\begin{aligned} C_d^{\Delta} &\cong \text{conv}\{e_1, \ldots, e_d, 1 - e_i, \ldots, 1 - e_d\} \\ &= \text{conv}\left\{\sum_{i \in A} e_i : |A| \in \{1, d-1\}\right\}. \end{aligned}$$

This yields

$$f(d) \;\geq\; f'(d) \geq 2^d$$

for all d. That equality $f(d) = 2^d$ holds for $d \leq 4$ is checked by complete enumeration. Such an enumeration (not complete) provided also the example that proves $f(5) \geq 40$:

```
DIM = 5
CONV_SECTION

0 0 0 0 0

1 0 1 0 0
1 1 1 0 0
0 1 0 1 0
0 1 1 1 0
0 0 1 0 1
0 0 1 1 1
1 0 0 1 0
1 0 0 1 1
0 1 0 0 1
1 1 0 0 1

1 1 1 1 1

END
```

For larger d, it seems that the following construction yields polytopes with considerably more facets: define

$$S_d \quad := \quad \left\{ \sum_{i\in A} e_i \;\middle|\; \begin{array}{l} \text{either } |A| \in \{1, |d|-1\}, \\ \text{or } |A|>0 \text{ is even and } A \subseteq \{1,2,\dots,\lfloor\frac{d}{2}\rfloor\}, \\ \text{or } |A|>0 \text{ is even and } A \subseteq \{1,2,\dots,\lfloor\frac{d}{2}\rfloor\}, \end{array} \right\}$$

One can compute, for example, that S_{10} is a 10-dimensional 0/1-polytope with $10,829 = (2.531971631)^{10}$ facets. Taking an appropriate direct sum of copies of S_{10} and of $C_i^{\Delta'}$ we obtain the following.

Corollary 2.2. *For $d \geq 0$ one has*

$$f(d) \geq f'(d) \geq (10,829)^{\lfloor d/10 \rfloor} 2^{d \bmod 10}.$$

Thus $f(d) > (2.5)^d$ for all sufficiently large d.

Upper bounds for $f(d)$ can be obtained from a volume argument due to I. Bárány [13, p. 25, Problem 0.15*] that we slightly refine in the following.

Theorem 2.3. *The maximal number of facets $f(d)$ of a d-dimensional 0/1-polytope satisfies*

$$f(d) \;\leq\; d! - (d-1)! + 2(d-1), \qquad \text{for } d \geq 3.$$

Proof. Let P be a d-dimensional 0/1-polytope. We can obtain $\text{conv}\{0,1\}^d$ from P by successive addition of 0/1-vertices, thus destroying all but the "trivial" facets of P. However, whenever a facet F_i of P is "destroyed" a cone over F_i is added. This cone is a d-dimensional 0/1-polytope, whence its volume is at least $1/d!$. Since the process stops at the d-dimensional 0/1-cube with $2d$ facets and volume 1, we get that

$$f_{d-1}(P) \quad \leq \quad 2d + d!(1 - Vol(P)). \tag{1}$$

On the other hand, P can be triangulated without new vertices, say into t 0/1-simplices of dimension d. Each of these simplices has volume at least $1/d!$, hence

$$t \quad \leq \quad d!Vol(P)$$

Each simplex has $d+1$ facets. The dual graph of the triangulation is connected; it has t nodes, hence at least $t-1$ edges. From this we get that at least $2(t-1)$ simplex facets are between simplices, so at most $t(d+1) - 2(t-1)$ simplex facets are in the surface of P. Since each facet of P is a union of simplex facets, we obtain

$$f_{d-1}(P) \;\leq\; t(d-1) + 2$$

and hence

$$f_{d-1}(P) \;\leq\; 2 + (d-1)d!Vol(P). \tag{2}$$

Addition of inequality (2) to the $(d-1)$-fold of (1) cancels the summands that involve the volume; we obtain

$$df_{d-1}(P) \quad \leq \quad 2 + 2d(d-1) \;+\; (d-1)d!.$$

Division by d and rounding down the right-hand side (since the left hand side is integral) yields the result. ∎

FIGURE 1. The subset P of the $2^2 \times 2^4$-grid.

3. THE COMPLEXITY OF TWO-DIMENSIONAL SHADOWS

The fact that $H(\mathcal{P}_d)$, the maximal number of vertices on an increasing path, is exponential follows already from the fact that there are 0/1-polytopes with exponentially many vertices, such that any two vertices are adjacent. So, for any generic linear function there is an increasing path through all the vertices. For example put $P := \mathrm{conv}(V) \subseteq \mathbb{R}^{k^2}$, with

$$V := \{xx^t : x \in \{0,1\}^k,\ x_k = 1\}.$$

This yields a polytope of dimension $d < k^2$, with 2^{k-1} vertices, any two adjacent [Barvinok]. In fact, for any $yy^t, zz^t \in V$ we can find a linear function $x \longmapsto a^x$ such that $a^t y = a^t z = 0$, but $a^t x \neq 0$ for any $x \in \{0,1\}^k$ with $x_k = 1$ and $x \neq y, z$. Then we compute

$$\langle aa^t, xx^t \rangle := \sum_{i=1}^{k} \sum_{j=1}^{k} (aa^t)_{ij}(xx^t)_{ij} = \Big(\sum_{i=1}^{k} a_i x_i\Big)\Big(\sum_{j=1}^{k} a_j x_j\Big) = (a^t x)^2 \geq 0,$$

with equality if and only if $x = y$ or $x = z$.

3.1. A Lower bound for $H_{sh}(\mathcal{P}_d)$.

We give a proof for a lower bound on the number of extremal vertices in the two-dimensional shadow of a 0/1-polytope. It relies on a special projection of the d-cube C_d onto a regular grid. We will choose a suitable subset of the projected points that lies in convex position.

Let us consider the following projection $\pi\colon \mathbb{R}^d \to \mathbb{R}^2$ for $d = 3k$ and k a positive integer: The first k coordinates x_i are projected down to $\binom{0}{2^{i-1}}$ for $1 \leq i \leq k$. The remaining $2k$ coordinates x_i are projected to $\binom{2^{i-k-1}}{0}$ for $k+1 \leq i \leq d$. By π we obtain a bijection between the vertices of the d-cube and the vertices of a $2^k \times 2^{2k}$ integer grid G.

Now we take the subset of vertices of C_d which corresponds to the subset P of the grid with

$$P = \{(i, i^2) \,|\, 0 \leq i \leq 2^k - 1\} \cup \{((2^k - 1 - i), 2^{2k} - 3 - i^2) \,|\, 0 \leq i \leq 2^k - 1\} \subseteq G.$$

P is just the set of grid points of a standard parabola, together with a rotated copy (see figure 1). All these points really lie inside the grid, which we check by calculating $(2^k - 1)^2 = 2^{2k} - 2^{k+1} + 1 \leq 2^{2k} - 1$.

It is obvious that this subset yields a projection with all vertices being extremal, and since we have $|P| = 2 \cdot 2^k$ we have a lower bound for the maximal number of extremal vertices in the two dimensional projection of a d-dimensional 0/1-polytope $H_{sh}(\mathcal{P}_d) \geq 2 \cdot 2^k$. This bound may be refined either by using a sligthly less growing convex function instead of the parabola, or by simply using the fact that the least significant bit (LSB) in the bit representation of i resp. i^2 is the same and the second LSB of i^2 is always 0, which we can use to squeeze the number of bits needed to represent the parabola and its mirror image, given $d \geq 4$. This yields

Theorem 3.1. *The maximal number of extremal vertices $H_{sh}(\mathcal{P}_d)$ of the two-dimensional shadow of a d-dimensional 0/1-polytope is bounded from below by*

$$2^{\lfloor \frac{d+5}{3} \rfloor} \leq H_{sh}(\mathcal{P}_d)$$

for $d \geq 4$.

Dimension	Lower Bound	Construction	Upper Bound	no. of vertices
1	—	2	9	2
2	—	4	14	4
3	—	6	21	8
4	8	10	32	16
5	8	14	50	32
6	8	18	76	64
7	16	22	117	128
8	16	32	179	256
9	16	42	274	512
10	32	52	420	1024
11	32	≥ 65	643	2048

TABLE 1. A comparision of the lower bound as given by Theorem 3.1, an explicit construction given by the projection vectors $\binom{2^i}{2^{d-i-1}}$ for $i = 0, \ldots, d-1$. and the upper bound as given by Corollary 3.3, as well as the number of vertices.

We would like to mention a rather similar, although more indirect method to show the same asymptotic lower bound. For this, project C_d for $d = 2k$ now to a regular $2^k \times 2^k$-grid. Using Satz 4.1.9 of [11] we find a convex polygon with $\frac{12}{(2\pi)^{2/3}} n^{2/3} + O(n \log n)$ extremal vertices on the grid, where $n = 2^k$. However, due to the O-term this does not help in investigating explicit lower bounds for small values of d (see Table 3.1).

3.2. **An upper bound for $H_{sh}(\mathcal{P}_d)$.** We derive upper bounds for $H_{sh}(\mathcal{P}_d)$ by relating this to a problem on set systems.

A collection of sets $\mathcal{S} \subseteq 2^{[d]}$ is said to have property **(SYM)** if the pairs $(A \backslash B, B \backslash A)$ are distinct for all $A, B \in \mathcal{S}$ with $A \neq B$. We define

$$X(d) = \max\{|\mathcal{S}| : \mathcal{S} \subseteq 2^{[d]} \text{ satisfies SYM}\}.$$

We note that the projection of a 0/1 d-polytope is described by a collection of d points $\mathcal{P} = \{p_1, \ldots, p_d\}$ in the plane. If p_i is the image of the unit vector $e_i \in \mathbb{R}^d$, then the image of a general 0/1 vector with support S is $\mathcal{P}(S) = \sum_{i \in S} p_i$. This defines a collection of at most 2^d points

$$2^{\mathcal{P}} := \{\mathcal{P}(S) | S \subseteq [d]\}.$$

If $g(\mathcal{P}, d)$ is the largest number of points in $2^{\mathcal{P}}$ in convex position, then

$$H_{sh}(\mathcal{P}_d) = \max_{\mathcal{P}} g(\mathcal{P}, d).$$

For subsets $S_1, S_2 \subseteq [d]$, the vector joining $\mathcal{P}(S_1)$ and $\mathcal{P}(S_2)$ is

$$\mathcal{P}(S_2) - \mathcal{P}(S_1) = \mathcal{P}(S_2 \backslash S_1) - \mathcal{P}(S_1 \backslash S_2)$$

which corresponds to the ordered pair $(S_2 \backslash S_1, S_1 \backslash S_2)$, and at most two copies of such a vector can appear in any polygon with vertices in $2^{\mathcal{P}}$. In fact, taking just half the vertices of the polygon, we ensure that each vector joining pairs of vertices appears exactly once. Then the subsets corresponding to the vertices of the polygon satisfy (SYM). We have thus shown that, the functions $H_{sh}(\mathcal{P}_d)$ and $X(d)$ are related by

$$H_{sh}(\mathcal{P}_d)/2 \leq X(d).$$

Thus it suffices to find an upper bound for $X(d)$ in order to bound $H_{sh}(\mathcal{P}_d)$. We first establish the following simple bound for $X(d)$:

If $\mathcal{S} \subseteq 2^{[d]}$ satisfies (SYM) and $|\mathcal{S}| = k$, then $k(k-1) \leq 3^d$, since the total number of disjoint pairs of subsets (A, B) in $[d]$ is 3^d. Hence $X(d) \leq 2\,3^{d/2}$.

We improve this bound in the following result.

Theorem 3.2.

$$X(d) \quad \leq \quad 3\,2^{d\frac{\log 3}{\log 6}}.$$

Corollary 3.3.

$$H_{sh}(P_d) \quad \leq \quad 6\,2^{d\frac{\log 3}{\log 6}}.$$

Proof. Let $\mathcal{S} \subseteq 2^{[d]}$ be a collection of sets satisfying (SYM). For a k-subset $T \subseteq [d]$, let $N(T)$ be the number of pairs (A, B) with $A, B \in \mathcal{S}$ and $A \backslash B, B \backslash A \subseteq T$. Let $\bar{T} = [d] \backslash T$ be the complement of T and let $m = 2^{d-k}$. We count $N(T)$ by partitioning $\mathcal{S}$ into subcollections $\mathcal{S}_1, \ldots, \mathcal{S}_m$ in such a way that if $A, B \in \mathcal{S}_i$ then $A \cap \bar{T} = B \cap \bar{T}$. This implies that $A \backslash B, B \backslash A \subseteq T$. If $|\mathcal{S}_i| = d_i$, then

$$d_1(d_1 - 1) + \cdots + d_m(d_m - 1) = N(T) \leq 3^k$$

since the number of disjoint pairs of subsets in T is at most 3^k. Then, $|\mathcal{S}| = d_1 + \cdots + d_m$ is maximum when $d_1 = \cdots = d$. Substituting this in the above inequality and simplifying we get

$$d_1 \leq (3^k/2^{d-k} + 1/4)^{1/2} + 1/2,$$

and

$$|\mathcal{S}| \leq 2^{d-k} d_1 \leq (2^{d-k} 3^k + 2^{2(d-k-1)})^{1/2} + 2^{d-k-1} \leq 2^{(d-k)/2} 3^{k/2} + 2^{d-k}.$$

The right hand side is minimized when $3^k = 2^{d-k}$. Hence choosing $k = \lceil d \log 2 / \log 6 \rceil$ we get

$$|\mathcal{S}| \leq 3\,2^{d \log 3 / \log 6}$$

as desired.

□

We conjecture that $X(d) = O(2^{d/2})$. A lower bound of the order of $2^{d/2}$ can be constructed for $X(d)$ by relating this problem to the existence of certain *Sidon Sets*, sets of integers with pairwise distinct sums, in the following sense.

By associating a set $S \subseteq [d]$ with the number $1 + \sum_{i \in S} 2^{i-1}$, we get a one-to-one correspondence between the subsets of $[d]$ and the elements of $[2^d]$. Then, given a Sidon subset of $[2^d]$, the corresponding collection of sets in $[d]$ satisfy (SYM). A Sidon subset of $[2^d]$ of size $2^{d/2} - c2^{5d/16}$ has been constructed in [5].

While the lower bound for $X(d)$ does not reveal any further information on the shadow vertex problem, it is of interest in its own right.

REFERENCES

[1] N. AMENTA & G. M. ZIEGLER: *Deformed products and maximal shadows,* Preprint 502/1996, TU Berlin, March 1996, 32 pages.

[2] L. J. BILLERA & A. SARANGARAJAN: *All* 0-1 *polytopes are traveling salesman polytopes,* Technical Report No. 1086, Cornell University 1993, 16 pages, to appear in *Combinatorica.*

[3] K. H. BORGWARDT: *The Simplex Method. A Probabilistic Analysis,* Algorithms and Combinatorics **1**, Springer 1987.

[4] T. CHRISTOF, M. JÜNGER & G. REINELT: *A complete description of the traveling salesman polytope on 8 nodes, Operations Research Letters* **10** (1991), 497–500.

[5] P. ERDŐS & P. TURÁN: *On a problem of Sidon in additive number theory, and on some related problems, J. London Math. Soc.* **16** (1941), 212–215. Addendum *J. London Math. Soc.* **19** (1944), 208.

[6] M. GRÖTSCHEL & M. PADBERG: *Polyhedral Theory/Polyhedral Computations,* in: E.L. Lawler, J.K. Lenstra, A.H.G. Rinnoy Kan, D.B. Shmoys (eds.), "The Traveling Salesman Problem", Wiley 1988, 251-360.

[7] M. HENK, J. RICHTER-GEBERT & G. M. ZIEGLER: *Basic properties of convex polytopes,* Preprint 472/1995, TU Berlin, October 1995, 28 pages; CRC Handbook on "Discrete and Computational Geometry" (J. E. Goodman, J. O'Rourke, eds.), to appear.

[8] G. KALAI: *The number of faces of centrally-symmetric polytopes,* (Research Problem), *Graphs and Combinatorics* **5** (1989), 389–391.

[9] D. NADDEF: *The Hirsch conjecture is true for* $(0,1)$*-polytopes, Math. Programming* **45** (1989), 109–110.

[10] A. S. SCHULZ, R. WEISMANTEL & G. M. ZIEGLER: 0/1-*Integer programming: augmentation and optimization are equivalent,* in: "Algorithms — ESA '95" (P. Spirakis, ed.), *Springer Lecture Notes in Computer Science* **979** (1995), 473-483.

[11] T. THIELE: *Extremalprobleme für Punktmengen,* Diplomarbeit, Freie Universität Berlin 1991, 80 pages.

[12] T. THIELE: *Geometric Selection Problems and Hypergraphs,* Dissertation, FU Berlin 1995, 89 pages.

[13] G. M. ZIEGLER: *Lectures on Polytopes, Graduate Texts in Mathematics* **152**, Springer-Verlag, New York 1995; *Updates, corrections, and more* available at `http://www.math.tu-berlin.de/~ziegler`

ULRICH HUND, DEPT. MATHEMATICS MA 6-1, TECHNISCHE UNIVERSITÄT BERLIN, STRASSE DES 17. JUNI 136, 10623 BERLIN, GERMANY

E-mail address: `hund@math.tu-berlin.de`

JÜRGEN RICHTER-GEBERT, DEPT. MATHEMATICS MA 6-1, TECHNISCHE UNIVERSITÄT BERLIN, STRASSE DES 17. JUNI 136, 10623 BERLIN, GERMANY

E-mail address: `richter@math.tu-berlin.de`

A. SARANGARAJAN, COMBINATORIAL OPTIMIZATION, ZIB BERLIN, HEILBRONNER STR. 10, 10711 BERLIN, GERMANY

E-mail address: `sarangarajan@zib-berlin.de`

GÜNTER M. ZIEGLER, DEPT. MATHEMATICS MA 6-1, TECHNISCHE UNIVERSITÄT BERLIN, STRASSE DES 17. JUNI 136, 10623 BERLIN, GERMANY

E-mail address: `ziegler@math.tu-berlin.de`