BASIC PROPERTIES OF CONVEX POLYTOPES

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INTRODUCTION

Convex polytopes are fundamental geometric objects that have been investigated since antiquity. The beauty of their theory is nowadays complemented by their importance for many other mathematical subjects, ranging from integration theory, algebraic topology and algebraic geometry (toric varieties) to linear and combinatorial optimization.

In this chapter we try to give a short introduction, provide a sketch of "what polytopes look like" and "how they behave," with many explicit examples, and briefly state some main results (where further details are in the subsequent chapters of this handbook). We concentrate on two main topics:

- Combinatorial properties: faces (vertices, edges, ..., facets) of polytopes and their relations, with special treatments of the classes of "lowdimensional polytopes" and "polytopes with few vertices;"
- Geometric properties: volume and surface area, mixed volumes and quermass-integrals, including explicit formulas for the cases of the regular simplices, cubes and crosspolytopes.

We refer to Grünbaum [16] for a comprehensive view of polytope theory, and to Ziegler [34] and Schneider [30] for recent treatments of the combinatorial resp. convex geometric aspects of polytope theory.

14.1 COMBINATORIAL STRUCTURE

GLOSSARY

$\mathcal{V}$-polytope: The convex hull of a finite set $X = \{x^1, \dots, x^n\}$ of points in $\mathbb{R}^d$:

$$P = \mathrm{conv}(X) \quad := \quad \Big\{ \sum_{i=1}^{n} \lambda_i x^i : \lambda_i \geq 0,\ \sum_{i=1}^{n} \lambda_i = 1 \Big\}.$$

$\mathcal{H}$-polytope: A bounded solution set of a finite system of linear inequalities:

$$P = P(A, b) \quad := \quad \big\{ x \in \mathbb{R}^d : a_i^T x \leq b_i \text{ for } 1 \leq i \leq m \big\},$$

where $A \in \mathbb{R}^{m \times d}$ is a real matrix with rows a_i^T, and $b \in \mathbb{R}^m$ is a real vector with entries b_i. Here boundedness means that there is a constant N such that $||x|| \leq N$ holds for all $x \in P$.

Polytope: A subset $P \subseteq \mathbb{R}^d$ that can be presented as a $\mathcal{V}$-polytope or (equivalently, by the Main Theorem below!) as an $\mathcal{H}$-polytope.

Dimension: The dimension of an arbitrary subset $S \subseteq \mathbb{R}^d$ is defined as the dimension of its affine hull: $\dim(S) := \dim(\text{aff}(S))$.

(Recall that aff(S), the affine hull of a set S, is $\{\sum_{j=1}^{p} \lambda_{i_j} x^{i_j} : x^{i_1}, ..., x^{i_p} \in S, \sum_{j=1}^{p} \lambda_{i_j} = 1\}$, the smallest affine subspace of $\mathbb{R}^d$ containing S.)

d-Polytope: A d-dimensional polytope. In what follows, a subscript in the name of a polytope usually denotes its dimension.

Interior and relative interior: The ***interior*** int(P) is the set of all points $x \in P$ such that for some $\varepsilon > 0$, the ε-ball $B_\varepsilon(x)$ around x is contained in P.

Similarly, the ***relative interior*** relint(P) is the set of all points $x \in P$ such that for some $\varepsilon > 0$, the intersection $B_\varepsilon(x) \cap \text{aff}(P)$ is contained in P.

Affine equivalence: For polytopes $P \subseteq \mathbb{R}^d$ and $Q \subseteq \mathbb{R}^e$, an affine map π: $\mathbb{R}^d \longrightarrow \mathbb{R}^e$, $x \longmapsto Ax + b$ that such that π maps P bijectively to Q. Note that π need not be injective or surjective. However, it has to restrict to a bijective map aff$(P) \longrightarrow$ aff(Q). In particular, if P and Q are affinely equivalent, then they have the same dimension.

THEOREM (Main Theorem of Polytope Theory (Minkowski, Weyl,. . .))
The definitions of $\mathcal{V}$-polytopes and of $\mathcal{H}$-polytopes are equivalent. That is, every $\mathcal{V}$-polytope has a description by a finite system of inequalities, and every $\mathcal{H}$-polytope can be obtained as the convex hull of a finite set of points (its vertices*).*

Geometrically, a $\mathcal{V}$-polytope is the projection of an $(n-1)$-dimensional simplex, while an $\mathcal{H}$-polytope is the bounded intersection of m closed halfspaces [34, Lect. 1]. To see the Main Theorem at work, consider the following two statements: the first one is easy to see for $\mathcal{V}$-polytopes, but not for $\mathcal{H}$-polytopes, and for the second statement we have the opposite effect.

1. ***Projections:*** Every image of a polytope P under an affine map π:$x \mapsto Ax+b$ is a polytope.

2. ***Intersections:*** Any intersection of a polytope with an affine subspace is a polytope.

However, the computational step from one of the Main Theorem's descriptions of polytopes to the other — a "convex hull computation" — is far from trivial. Essentially, there are three types of algorithms available: inductive algorithms (inserting vertices, using a so-called beneath-beyond technique), projection resp. intersection algorithms (known as Fourier-Motzkin elimination resp. double description algorithms), and reverse search methods (as introduced by Avis & Fukuda). For explicit computations one can use public domain codes such as the PORTA code [12] that we use here, which implements an algorithm of the second type.

In the following definitions of d-simplices, d-cubes and d-crosspolytopes we give both a $\mathcal{V}$- and an $\mathcal{H}$-presentation in each case. From this one can see that the $\mathcal{H}$-presentation can have exponential "size" in terms of the size of the $\mathcal{V}$-presentation (e.g., for the d-crosspolytopes), and vice versa (for the d-cubes).

DEFINITION (d-Simplex) A (regular) d-dimensional simplex in $\mathbb{R}^d$ is given by

$$\begin{aligned} T_d &:= \operatorname{conv}\{e^1, e^2, \ldots, e^d, \frac{1-\sqrt{d+1}}{d}(e^1 + \ldots + e^d)\} \\ &= \Big\{x \in \mathbb{R}^d : \sum_{i=1}^d x_i \le 1, \quad -(1+\sqrt{d+1}+d)x_k + \sum_{i=1}^d x_i \le 1 \text{ for } 1 \le k \le d\Big\}, \end{aligned}$$

where $e^1, \ldots, e^d$ denotes the coordinate unit vectors in $\mathbb{R}^d$.

The simplices T_d are ***regular polytopes*** (with a symmetry group that is flag-transitive — see Chapter 17): the parameters have been chosen so that all edges of T_d have length $\sqrt{2}$. Furthermore, the origin $0 \in \mathbb{R}^d$ is in the interior of T_d: this is clear from the $\mathcal{H}$-presentation.

However, for the combinatorial theory one considers polytopes that differ only by a change of coordinates (an affine transformation) to be equivalent. Thus, we would refer to any d-polytope that can be presented as the convex hull of $d+1$ points as a *d-simplex*, since any two such polytopes are equivalent with respect to an affine map. Other standard choices include

$$\begin{aligned} \Delta_d &:= \operatorname{conv}\{0, e^1, e^2, \ldots, e^d\} \\ &= \Big\{x \in \mathbb{R}^d : \sum_{i=1}^d x_i \le 1, \quad x_k \ge 0 \text{ for } 1 \le k \le d\Big\}. \end{aligned}$$

and the $(d-1)$-dimensional simplex in $\mathbb{R}^d$ given by

$$\begin{aligned} \Delta'_{d-1} &:= \operatorname{conv}\{e^1, e^2, \ldots, e^d\} \\ &= \Big\{x \in \mathbb{R}^d : \sum_{i=1}^d x_i = 1, \quad x_k \ge 0 \text{ for } 1 \le k \le d\Big\}. \end{aligned}$$

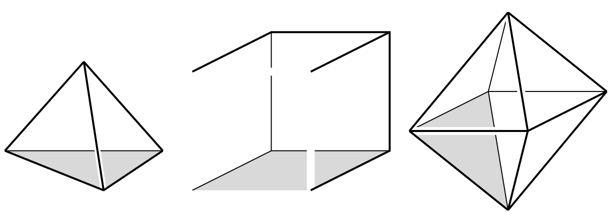


Figure 14.1
A 3-simplex, a 3-cube, and a 3-dimensional crosspolytope (octahedron).

DEFINITION (d-Cube and d-Crosspolytope) A *d-cube* (a.k.a. the *d-dimensional hypercube*) is

$$\begin{aligned} C_d &:= \operatorname{conv}\{\alpha_1 e^1 + \alpha_2 e^2 + \ldots + \alpha_d e^d : \alpha_1, \ldots, \alpha_d \in \{+1, -1\}\} \\ &= \Big\{x \in \mathbb{R}^d : -1 \le x_k \le 1 \text{ for } 1 \le k \le d\Big\}, \end{aligned}$$

and a d-dimensional *crosspolytope* in $\mathbb{R}^d$ (known as the *octahedron* for $d = 3$) is given by

$$C_d^{\Delta} \quad := \quad \text{conv}\{\pm e^1, \pm e^2, \ldots, \pm e^d\} \quad = \quad \Big\{x \in \mathbb{R}^d : \sum_{i=1}^{d} |x_i| \le 1\Big\}.$$

Again, there are other very natural choices, among them

$$\begin{aligned} [0,1]^d &= \text{conv}\{\sum_{i \in S} e^i : S \subseteq \{1, 2, \ldots, d\}\} \\ &= \Big\{x \in \mathbb{R}^d : 0 \le x_k \le 1 \text{ for } 1 \le k \le d\Big\}, \end{aligned}$$

the d-dimensional ***unit cube***.

As another example to illustrate concepts and results we will occasionally use the poor little unnamed polytope with six vertices shown in Figure 14.2.

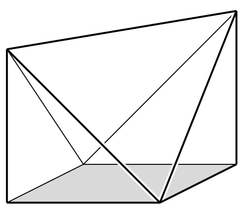


Figure 14.2
Our unnamed "typical" 3-polytope. It has 6 vertices, 11 edges and 7 facets.

This polytope without a name can be presented as a $\mathcal{V}$-polytope by listing its six vertices. The following coordinates make it into a subpolytope of the 3-cube C_3: the vertex set consists of all but two vertices of C_3. Our list below (on the left) is in the format used as input for the PORTA program [12], e.g. in a file named `unnamedpoly.poi`. From these data the PORTA program produces a description (on the right) of the polytope as an $\mathcal{H}$-polytope, stored in the file `unnamedpoly.poi.ieq`

```
DIM = 3
CONV_SECTION
( 1)   1  1  1
( 2)  -1 -1  1
( 3)   1  1 -1
( 4)   1 -1 -1
( 5)  -1  1 -1
( 6)  -1 -1 -1
END
```

```
INEQUALITIES_SECTION
(  1)     +x2     <= 1
(  2) +x1         <= 1
(  3) -x1         <= 1
(  4)     -x2     <= 1
(  5)         -x3 <= 1
(  6) -x1+x2+x3 <= 1
(  7) +x1-x2+x3 <= 1
```

Unbounded polyhedra can, via projective transformations, be treated as polytopes with a distinguished facet (see [34, p. 75]). In this respect, we do not lose anything on the combinatorial level if we restrict the following discussion to the setting of full-dimensional convex polytopes: d-polytopes embedded in $\mathbb{R}^d$.

14.1.1 FACES

GLOSSARY

Support function: Given a polytope $P \subseteq \mathbb{R}^d$, the function

$$h(P,\cdot)\colon \mathbb{R}^d \to \mathbb{R}, \qquad h(P,x) := \sup\{\langle x,y\rangle : y \in P\},$$

where $\langle x,y\rangle$ denotes the inner product on $\mathbb{R}^d$. (Since P is compact one may replace sup by max.)

Supporting hyperplane: For $v \in \mathbb{R}^d \setminus \{0\}$ the hyperplane

$$H(P,v) := \{x \in \mathbb{R}^d : \langle x,v\rangle = h(P,v)\}$$

is the ***supporting hyperplane*** of P with ***outer normal vector*** v. Note that $H(P,\mu v) = H(P,v)$ for $\mu \in \mathbb{R}$, $\mu > 0$. For a vector u of the $(d-1)$-dimensional ***unit sphere*** S^{d-1}, $h(P,u)$ is the signed distance of the supporting plane $H(P,u)$ from the origin. (For $v = 0$ we set $H(P,0) := \mathbb{R}^d$, which is not a hyperplane.)

Face: The intersection of P with a supporting hyperplane $H(P,v)$ is called a ***(nontrivial) face***, or more precisely a ***k-face*** if the dimension of $\mathrm{aff}(P \cap H(P,v))$ is k. Each face is itself a polytope.

The set of all k-faces is denoted by $\mathcal{F}_k(P)$ and its cardinality by $f_k(P)$.

f-Vector: The vector of face numbers $\boldsymbol{f}(P) = (f_0(P), f_1(P), \ldots, f_{d-1}(P))$ associated to a d-polytope.

Trivial faces: The empty set $\emptyset$ and the polytope P itself are considered ***trivial faces*** of P, of dimensions -1 and $\dim(P)$, respectively. All faces other than P are ***proper faces***.

Vertex, edge, facet: The faces of dimension 0 and 1 are called ***vertices*** and ***edges***, respectively. The $(\dim(P)-1)$-faces of P are called ***facets***.

Facet-vertex incidence matrix: The matrix $M \in \{0,1\}^{f_{d-1}(P)\times f_0(P)}$ which has an entry $M(F,v) = 1$ if the facet F contains the vertex v, and $M(F,v) = 0$ otherwise.

Graded poset: A partially ordered set $(P,\leq)$ with a unique minimal element $\hat{0}$, a unique maximal element $\hat{1}$, and a ***rank function*** $r\colon P \longrightarrow \mathbb{N}_0$ that satisfies
(1) $r(\hat{0}) = 0$, and $p < p'$ implies $r(p) < r(p')$, and
(2) $p < p'$ and $r(p') - r(p) > 1$ implies that there is a $p'' \in P$ with $p < p'' < p'$.

Lattice L: A partially ordered set $(P,\leq)$ in which every pair of elements $p, p' \in P$ has a unique maximal lower bound, called the ***meet*** $p \wedge p'$, and a unique minimal upper bound, called the ***join*** $p \vee p'$.

Atom, coatom: If L is a graded lattice, the minimal elements of $L \setminus \{\hat{0}\}$ (i.e., the elements of rank 1) are the atoms of L. Similarly, the maximal elements of $L \setminus \{\hat{1}\}$ (i.e., the elements of rank $r(\hat{1})-1$) are the coatoms of L.
A graded lattice is ***atomic*** if every element is a join of a set of atoms, and it is ***coatomic*** if every element is a meet of set of coatoms.

Face lattice L(P): The set of all faces of P, partially ordered by inclusion.

Combinatorially isomorphic: Polytopes whose face lattices are isomorphic as abstract (unlabelled) partially ordered sets/lattices.

Equivalently, P and P' are combinatorially equivalent if their facet-vertex incidence matrices differ only by column and row permutations.

Combinatorial type: An equivalence class of polytopes under combinatorial equivalence.

THEOREM (Face lattices of polytopes) *The face lattices of convex polytopes are finite, graded, atomic and coatomic lattices.*

The meet operation $G \wedge H$ is given by intersection, while the join $G \vee H$ is the intersection of all facets that contain both G and H. The rank function on $L(P)$ is given by $r(G) = \dim(G){+}1$.

The minimal nonempty faces of a polytope are its vertices: they correspond to atoms of the lattice $L(P)$. Every face is the join of its vertices, hence $L(P)$ is atomic. Similarly, the maximal proper faces of a polytope are its facets: they correspond to the coatoms of $L(P)$. Every face is the intersection of the facets it is contained in, hence face lattices of polytopes are coatomic.

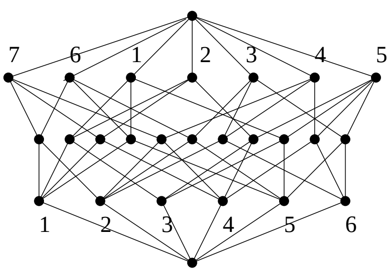


Figure 14.3
The face lattice of our unnamed 3-polytope. The 7 coatoms and the 6 atoms have been labeled to correspond to the labels in the facet-vertex incidence matrix. Thus, the downwards-path from the coatom "4" to the atom "2" represents the fact that the facet numbered (4) *contains the vertex* (2).

The face lattice is a complete encoding of the combinatorial structure of a polytope. However, in general the encoding by a facet-vertex incidence matrix is more efficient. The following matrix — also provided by PORTA — represents our little unnamed 3-polytope:

$$
M \;=\; \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array}
\begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\
\begin{pmatrix}
1 & 0 & 1 & 0 & 1 & 0 \\
1 & 0 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 1 \\
0 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 1 & 1 & 1 \\
1 & 1 & 0 & 0 & 1 & 0 \\
1 & 1 & 0 & 1 & 0 & 0
\end{pmatrix} \end{array}
$$

How do we decide whether a set of vertices $\{v^1, \ldots, v^k\}$ is (the vertex set of) a face of P? This is the case if and only if no other vertex v^0 is contained in all the facets that contain $\{v^1, \ldots, v^k\}$. This criterion makes it possible, for example, to derive the edges of a polytope P from a facet-vertex matrix.

For low-dimensional polytopes, the criterion can be simplified: if $d \le 4$, then two vertices are connected by an edge if and only if there are at least $d-1$ different facets that contain them both. However, the same is not true any longer for 5-dimensional polytopes, where vertices may be non-adjacent despite being contained in many common facets. (The best way to see this is by using polarity; see below.)

14.1.2 POLARITY

GLOSSARY

Polarity: If $P \subseteq \mathbb{R}^d$ is a d-polytope with the origin in its interior, then the *polar* of P is the d-polytope

$$P^{\Delta} \quad := \quad \{y \in \mathbb{R}^d : \langle y, x\rangle \le 1 \text{ for all } x \in P\}.$$

Stellar subdivision: The stellar subdivision of a polytope P in a face F is the polytope $\mathrm{conv}(P \cup x^F)$, where x^F is a point of the form $y^F - \varepsilon(y^P - y^F)$, where y^P is in the interior of P, y^F is in the relative interior of F, and ε is small enough.

Vertex figure P/v: If v is a vertex of P, then $P/v := P \cap H$ is the polytope obtained by intersecting P with a hyperplane H that has v on one side and all the other vertices of P on the other side.

Cutting off a vertex: The polytope $P \cap H^-$ obtained by intersecting P with a closed halfspace H^- that does not contain the vertex v, but contains all other vertices of P in its interior. (In this situation, $P \cap H^+$ is a pyramid over the vertex figure P/v.)

Quotient of P: A polytope obtained from P by taking vertex figures (possibly) several times.

Simplicial polytope: A polytope all of whose facets (equivalently, proper faces) are simplices.

Simple polytope: A polytope all of whose vertex figures (equivalently, proper quotients) are simplices.

Polarity is a fundamental construction in the theory of polytopes. One always has $P^{\Delta\Delta} = P$, under the assumption that P has the origin in its interior. This condition can always be obtained after a change of coordinates. In particular, we speak of (combinatorial) polarity between d-polytopes Q and R that are combinatorially isomorphic to P and P^{Δ}, respectively.

Any $\mathcal{V}$-presentation of P yields an $\mathcal{H}$-presentation of P^{Δ}, and conversely, via

$$P = \mathrm{conv}\{v^1, \dots, v^n\} \quad \Longleftrightarrow \quad P^{\Delta} = \{x \in \mathbb{R}^d : \langle v^i, x\rangle \le 1 \text{ for } 1 \le i \le n\}.$$

There are basic relations between polytopes and polytopal constructions under polarity. For example, the fact that the d-crosspolytopes C_d^{Δ} are the polars of the d-cubes C_d is built into our notation. More generally, the polars of simple polytopes are simplicial, and conversely. This can be deduced from the fact that the facets F of a polytope P correspond to the vertex figures P^{Δ}/v of its polar P^{Δ}. In fact,

F and P^Δ/v are combinatorially polar in this situation. More generally, one has a correspondence between faces and quotients under polarity.

At a combinatorial level, all this can be derived from the fact that the face lattices $L(P)$ and $L(P^\Delta)$ are anti-isomorphic: $L(P^\Delta)$ may be obtained from $L(P)$ by reversing the order relations. Thus, lower intervals in $L(P)$, corresponding to faces of P, translate under polarity into upper intervals of $L(P^\Delta)$, corresponding to quotients of P^Δ.

14.1.3 BASIC CONSTRUCTIONS

GLOSSARY

For the following constructions, let

$P \subseteq \mathbb{R}^d$ be a d-dimensional polytope with n vertices and m facets, and

$P' \subseteq \mathbb{R}^{d'}$ a d'-dimensional polytope with n' vertices and m' facets.

Scalar multiple: For $\lambda \in \mathbb{R}$, the scalar multiple λP is defined by $\lambda P := \{\lambda x : x \in P\}$. P and λP are combinatorially (in fact, affinely) isomorphic for all $\lambda \neq 0$. In particular, $(-1)P = -P = \{-p : p \in P\}$, and $(+1)P = P$.

Minkowski sum: $P + P' := \{p + p' : p \in P, p' \in P'\}$.

It is also useful to define the difference as $P - P' = P + (-P')$. The polytopes $P + \lambda P'$ are combinatorially isomorphic for all $\lambda > 0$, and similarly for $\lambda < 0$.

If $P' = \{p'\}$ is one single point, then $P - \{p'\}$ is the image of P under the translation that takes p' to the origin.

Product: The $(d+d')$-dimensional polytope

$P \times P' := \{(p, p') \in \mathbb{R}^{d+d'} : p \in P, p' \in P'\}$.

$P \times P'$ has $n \cdot n'$ vertices and $m + m'$ facets.

Join: The convex hull $P * P'$ of $P \cup P'$, after embedding P and P' in a space where their affine hulls are skew. For example,

$P * P' := \text{conv}(\{(p, 0, 0) \in \mathbb{R}^{d+d'+1} : p \in P\} \cup \{(0, p', 1) \in \mathbb{R}^{d+d'+1} : p' \in P'\})$.

$P*P'$ has dimension $d+d'+1$ and $n+n'$ vertices. Its k-faces are the joins of i-faces of P and $(k-i-1)$-faces of P', hence $f_k(P * P') = \sum_{i=-1}^{k} f_i(P) f_{k-i-1}(P')$.

Free sum: The free sum is the $(d+d')$-dimensional polytope

$P \oplus P' := \text{conv}(\{(p, 0) \in \mathbb{R}^{d+d'} : p \in P\} \cup \{(0, p') \in \mathbb{R}^{d+d'} : p' \in P'\})$.

Thus the free sum $P \oplus P'$ is a projection of the join $P * P'$. If both P and P' have the origin in their interiors — this is the "usual" situation for creating free sums —, then $P \oplus P'$ has $n + n'$ vertices and $m \cdot m'$ facets.

Pyramid: The join $\text{pyr}(P) := P * \{0\}$ of P with a point (a 0-dimensinal polytope $P' = \{0\} \subseteq \mathbb{R}^0$). The pyramid $\text{pyr}(P)$ has $n + 1$ vertices and $m + 1$ facets.

Prism: The product $\text{prism}(P) := P \times I$, where I denotes the real interval $I = [-1, +1] \subseteq \mathbb{R}$.

Bipyramid: If P has the origin in its interior, then the bipyramid over P is the $(d+1)$-dimensional polytope constructed as the free sum $\text{bipyr}(P) := P \oplus I$.

Lawrence extension: If $p \in \mathbb{R}^d$ is a point outside P, then the free sum $(P - \{p\}) \oplus [1, 2]$ is a *Lawrence extension of P at p*. (For $p \in P$ this is just a pyramid.)

Of course, the many constructions listed in the glossary above are not independent of each other. For instance, some of these constructions are related by polarity: for polytopes P and P' with the origin in their interiors, the product and the free sum constructions are related by polarity,

$$P \times P' \;=\; (P^{\Delta} \oplus P'^{\Delta})^{\Delta},$$

and this specializes to polarity relations among the pyramid, bipyramid and prism constructions,

$$\mathrm{pyr}(P) \;=\; (\mathrm{pyr}(P^{\Delta}))^{\Delta} \qquad \text{and} \qquad \mathrm{prism}(P) \;=\; (\mathrm{bipyr}(P^{\Delta}))^{\Delta}.$$

Similarly, "cutting off a vertex" is polar to "stellar subdivision in a facet."

It is interesting to study — and this has not really been done systematically — how the basic operations on polytopes generate complicated convex polytopes from simpler ones. For example, starting from a one-dimensional polytope $I = C_1 = [-1,+1] \subset \mathbb{R}$, the direct product construction generates the cubes C_d, while free sums generate the crosspolytopes C_d^{Δ}.

Even more complicated centrally symmetric polytopes, the *Hanner polytopes*, are obtained from copies of the interval I by using sums and free sums. They are interesting since they achieve with equality the conjectured bound that all centrally symmetric d-polytopes have at least 3^d nonempty faces (Kalai [20]).

Every polytope can be viewed as a region of a hyperplane arrangement: for this, take as $\mathcal{A}_P$ the set of all hyperplanes of the form $\mathrm{aff}(F)$, where F is a facet of P. For additional points, such as the points outside the polytope used for Lawrence extensions, or those used for stellar subdivisions, it is often important only in which region, or in which lower-dimensional region, of the arrangement $\mathcal{A}_P$ they lie.

The Lawrence extension, by the way, may seem like quite a harmless little construction. However, it has the amazing property that it can encode the structure of a point *outside* a d-polytope into the boundary structure of a $(d{+}1)$-polytope. This accounts for a large part of the "special" 4- and 5-polytopes in the literature, such as the 4-polytopes for which a facet, or even a 2-face, cannot be prescribed in shape [26].

14.1.4 MORE EXAMPLES

There are many interesting classes of polytopes arising from diverse areas of mathematics (as well as physics, optimization, crystallography, etc.). Some of these are discussed below. You will find many more classes of examples discussed in other chapters of this handbook. For example, regular and semiregular polytopes are discussed in Chapter 17, while polytopes that arise as Voronoï cells of lattices appear in Chapter 8.

GLOSSARY

Graph of a polytope: The graph $G(P) = (V(P), E(P))$ with vertex set $V(P) = \mathcal{F}_0(P)$ and edge set $E(P) = \{\{v^1, v^2\} \subseteq \binom{V}{2} : \mathrm{conv}\{v^1, v^2\} \in \mathcal{F}_1(P)\}$.

Zonotope: Any polytope Z that can be represented as the image of an n-dimensional cube C_n under an affine map; equivalently, any polytope that can be written as a Minkowski sum of n line segments (1-dimensional polytopes). The smallest n such that Z is an image of C_n is the ***number of zones*** of Z.

Moment curve: The curve γ in $\mathbb{R}^d$ defined by $\gamma:\mathbb{R} \longrightarrow \mathbb{R}^d$, $t \longmapsto (t, t^2, \ldots, t^d)^T$.

Cyclic polytope: The convex hull of a finite set of points on a moment curve, or any polytope combinatorially equivalent to it.

k-Neighborly polytope: A polytope such that each subset of at most k vertices forms the vertex set of a face. Thus every polytope is 1-neighborly, and a polytope is 2-neighborly if and only if its graph is complete.

Neighborly polytope: A d-dimensional polytope that is $\lfloor d/2 \rfloor$-neighborly.

0/1-Polytope: A polytope all of whose vertex coordinates are 0 or 1, that is, whose vertex set is a subset of the vertex set $\{0,1\}^d$ of the unit cube.

Zonotopes

Zonotopes appear in quite different disguises. They can equivalently be defined as the Minkowski sums of finite sets of line segments (1-dimensional polytopes), as the affine projections of d-cubes, or as polytopes all of whose faces (equivalently, all 2-faces) exhibit central symmetry. Thus a 2-dimensional polytope is a zonotope if and only if it is centrally symmetric.

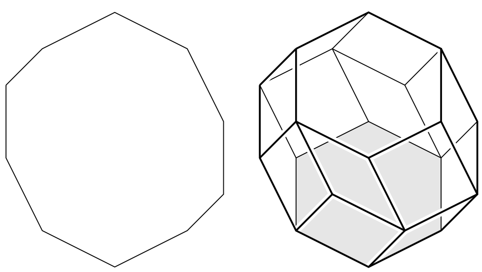


Figure 14.4
A 2-dimensional and a 3-dimensional zonotope, each with 5 zones. (The 2-dimensional one is a projection of the 3-dimensional one; note that every projection of a zonotope is a zonotope.)

Among the most prominent zonotopes are the permutahedra: The permutahedron Π_{d-1} is constructed by taking the convex hull of all d-vectors whose coordinates are $\{1, 2, \ldots, d\}$, in any order. The permutahedron Π_{d-1} is a $(d-1)$-dimensional polytope (contained in the hyperplane $\{x \in \mathbb{R}^d : \sum_{i=1}^d x_i = d(d+1)/2\}$) with $d!$ vertices and $2^d - 2$ facets.

One unusual feature of permutahedra is that they are simple zonotopes: these are rare in general, and the (unsolved) problem of classifying them is equivalent to the problem of classifying all simplicial arrangements of hyperplanes (see Section 7.3.3).

Zonotopes are important because their theory is equivalent to the theories of vector configurations (realizable oriented matroids) and of hyperplane arrangements. In fact, the system of line segments that generates a zonotope can be considered as a vector configuration, and the hyperplanes that are orthogonal to

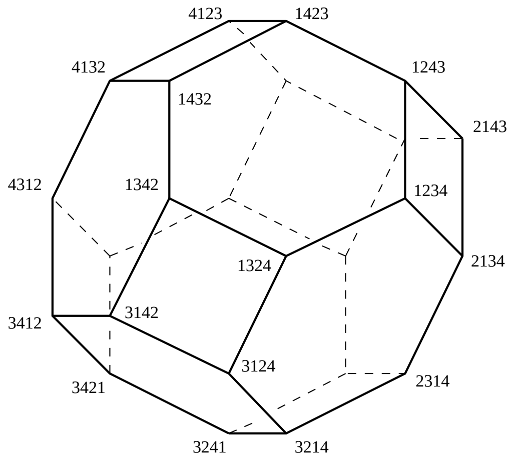


Figure 14.5
The 3-dimensional permutahedron Π_3. (The vertices are labeled by the permutations that, when applied to the coordinate vector in $\mathbb{R}^4$, yield $(1,2,3,4)^T$.)

the line segments provide the associated hyperplane arrangement. We refer to [6, Sect. 2.2] and [34, Lect. 7].

Finally, we mention in passing a surprising bijective correspondence between the tilings of a zonotope with smaller zonotopes and oriented matroid liftings (realizable or not) of the oriented matroid of a zonotope. This correspondence is known as the *Bohne-Dress Theorem*; we refer to Richter-Gebert & Ziegler [27].

Cyclic Polytopes

Cyclic polytopes can be constructed by taking the convex hull of $n > d$ points on the moment curve in $\mathbb{R}^d$. The "standard construction" is to define a cyclic polytope $C_d(n)$ as the convex hull of n integer points on this curve, such as

$$C_d(n) \ := \ \operatorname{conv}\{\gamma(1), \gamma(2), \ldots, \gamma(n)\}.$$

However, the combinatorial type of $C_d(n)$ is given by the — entirely combinatorial — ***Gale evenness criterion***: If $C_d(n) = \operatorname{conv}\{\gamma(t_1), \ldots, \gamma(t_n)\}$, with $t_1 < \ldots < t_n$, then $\gamma(t_{i_1}), \ldots, \gamma(t_{i_d})$ determine a facet if and only if the number of indices in $\{i_1, ..., i_d\}$ lying between any two indices *not* in that set is even. Thus, the combinatorial type does not depend on the specific choice of points on the moment curve [34, Example 0.6; Thm. 0.7].

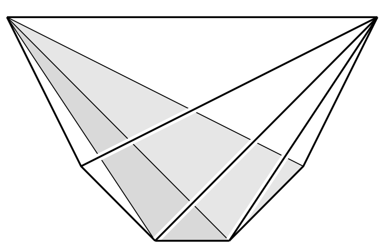


Figure 14.6
A 3-dimensional cyclic polytope $C_3(6)$ with 6 vertices. (In a projection of γ to the x_1x_2-plane, the curve γ and hence the vertices of $C_3(6)$ lie on the parabola $x_2 = x_1^2$.)

The first property of cyclic polytopes to notice is that they are simplicial. The second, more surprising, property is that they are neighborly. This implies that among all d-polytopes P with n vertices, the cyclic polytopes maximize the number $f_i(P)$ of i-dimensional faces for $i < \lfloor d/2 \rfloor$. The same fact holds for all i: this is part of McMullen's Upper Bound Theorem, see below. In particular, cyclic polytopes have a very large number of facets,

$$f_{d-1}(C_d(n)) \quad = \quad \binom{n - \lceil \frac{d}{2} \rceil}{\lfloor \frac{d}{2} \rfloor} \; + \; \binom{n - 1 - \lceil \frac{d-1}{2} \rceil}{\lfloor \frac{d-1}{2} \rfloor}.$$

For example, we get that a cyclic 4-polytope $C_4(n)$ has $n(n-3)/2$ facets. Thus $C_4(8)$ has 8 vertices, any two of them adjacent, and 20 facets (this is more than the 16 facets of the 4-dimensional crosspolytope, which also has 8 vertices!).

Neighborly Polytopes

Here are a few observations about neighborly polytopes. For more information, see [6, Sect. 9.4] and the references quoted there.

The first observation is that if a polytope is k-neighborly for some $k > \lfloor d/2 \rfloor$, then it is a simplex. Thus, if one ignores the simplices, then $\lfloor d/2 \rfloor$-neighborly polytopes form the extreme case, which motivates calling them simply "neighborly." However, only in even dimensions $d = 2m$ do the neighborly polytopes have very special structure. For example, one can show that even-dimensional neighborly polytopes are necessarily simplicial, but this is not true in general. For the latter, note that, for example, all 3-dimensional polytopes are neighborly by definition, and that if P is a neighborly polytope of dimension $d = 2m$, then $\mathrm{pyr}(P)$ is neighborly of dimension $2m{+}1$.

All simplicial neighborly d-polytopes with n vertices have the same number of facets (in fact, the same f-vector $(f_0, f_1, \ldots, f_{d-1})$) as $C_d(n)$. They constitute the class of polytopes with the maximal number of i-faces for all i: this is the statement of McMullen's Upper Bound Theorem. We refer to Chapter 16 for a thorough discussion of f-vector theory.

For $n \leq d{+}3$ every neighborly polytope is combinatorially isomorphic to a cyclic polytope. (This covers, for instance, the polar of the product of two triangles, $(\Delta_2 \times \Delta_2)^\Delta$, which is easily seen to be a 4-dimensional neighborly polytope with 6 vertices; see Figure 14.9.) The first example of an even-dimensional neighborly polytope that is not cyclic appears for $d = 4$ and $n = 8$. It can easily be described in terms of its affine Gale diagram; see below.

Neighborly polytopes may at first glance seem to be very peculiar and rare objects, but there are several indications that they are not quite as unusual as they seem. In fact, the class of neighborly polytopes is believed to be very rich. Thus, Shemer [29] has shown that for fixed even d the number of non-isomorphic neighborly d-polytopes with n vertices grows superexponentially with n. Also, many of the 0/1-polytopes studied in combinatorial optimization turn out to be at least 2-neighborly. Both these effects illustrate that "neighborliness" is not an isolated phenomenon.

Three Problems

1. Can every neighborly d-polytope $P \subseteq \mathbb{R}^d$ with n vertices be extended by a new vertex $v \in \mathbb{R}^d$ to a neighborly polytope $P' := \text{conv}(P \cup \{v\})$ with $n+1$ vertices? [29, p. 314]
2. It is a classic problem of Perles whether every simplicial polytope is a quotient of a neighborly polytope. (For polytopes with at most $d+4$ vertices this was recently confirmed by Hund [19].)
3. In some models of random polytopes is seems that
   - one obtains a neighborly polytope with high probability (which increases rapidly with the dimension of the space),
   - the most probable combinatorial type is a cyclic polytope,
   - but still this probability of a cyclic polytope tends to zero.

   However, none of this has been proved. (See Bokowski & Sturmfels [10, p. 101] and Bokowski, Richter-Gebert & Schindler [7].)

0/1-Polytopes

There is a 0/1 polytope (given in terms of a $\mathcal{V}$-presentation) associated with every finite set system $\mathcal{S} \subseteq 2^E$ (where E is a finite set, and 2^E denotes the collection of all of its subsets), via

$$P[\mathcal{S}] \quad := \quad \text{conv}\Big\{ \sum_{i \in F} e^i : F \in \mathcal{S} \Big\} \quad \subseteq \quad \mathbb{R}^E.$$

In combinatorial optimization, there is an extensive literature available on $\mathcal{H}$-presentations of special 0/1-polytopes, such as

- the ***traveling salesman polytopes*** T^n, where E is the edge set of a complete graph K_n, and $\mathcal{F}$ is the set of all $(n-1)!$ Hamilton cycles (simple circuits through all the vertices) in E (see Grötschel & Padberg [14]),
- the ***cut and equicut polytopes***, where E is again the edge set of a complete graph, and $\mathcal{S}$ represents, for example, the family of all cuts, or all equicuts, of the graph (see Deza & Laurent [13]).

Besides their importance for combinatorial optimization, there is a great deal of interesting polytope theory associated with such polytopes. For a striking example, see the equicut polytopes used by Kahn & Kalai [21] in their recent disproof of Borsuk's conjecture.

Despite the detailed structure theory for the "special" 0/1-polytopes of combinatorial optimization, there is very little known about "general" 0/1-polytopes. For example, what is the "typical," or the maximal, number of facets of a 0/1-polytope? What is the maximal number of faces in a 2-dimensional projection? (Such questions are not only intrinsically interesting, their answers might also provide new clues for basic questions of linear and combinatorial optimization.)

14.1.5 THREE-DIMENSIONAL POLYTOPES AND PLANAR GRAPHS

GLOSSARY

d-Connected graph: A connected graph that remains connected if any $d-1$ vertices are deleted.

Drawing of a graph: A representation in the plane where the vertices are represented by distinct points, and simple Jordan arcs are drawn between the pairs of adjacent vertices.

Planar graph: A graph that can be drawn in the plane with Jordan arcs that are disjoint except for their endpoints.

Realization space: The set of all coordinatizations of a combinatorial structure, modulo affine coordinate transformations. (See Chapter 7, Section 7.3.2).

Isotopy property: A combinatorial structure (such as a combinatorial type of polytope) has the *isotopy property* if any two realizations can be deformed into each other continuously, while maintaining the combinatorial type. Equivalently, the isotopy property holds for a combinatorial structure if and only if its realization space is connected.

THEOREM (Steinitz' Theorem [32]**)** *For every 3-dimensional polytope P, the graph $G(P)$ is a planar, 3-connected graph. Conversely, for every planar 3-connected graph there is a unique combinatorial type of 3-polytope P with $G(P) \cong G$.*

Furthermore, the realization space $\mathcal{R}(P)$ of a combinatorial type of 3-polytope is homeomorphic to $\mathbb{R}^{f_1(P)-6}$, and contains rational points. In particular, 3-dimensional polytopes have the isotopy property, and they can be realized with integer vertex coordinates.

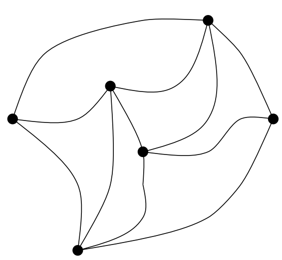


Figure 14.7
A (planar drawing of a) 3-connected, planar, unnamed graph. The formidable task of any proof of Steinitz' Theorem is to construct a 3-polytope with this graph.

There are two essentially different ways known to prove Steinitz' Theorem. The first one [32] provides a construction sequence for any type of 3-polytope, starting from a tetrahedron, and using only local operations such as cutting off vertices, and polarity. The second type of proof realizes any combinatorial type by a global minimization argument, which as an intermediate step provides a special planar representation of the graph by a framework with a positive self-stress [25, 33].

Two Problems

Because of Steinitz' Theorem and it extensions and corollaries, the theory of 3-dimensional polytopes is quite complete and satisfactory. Nevertheless, some basic open problems remain.

1. It can be shown that every combinatorial type of 3-polytope with n vertices can be realized with integer coordinates in $\{1, 2, \ldots, 43^n\}^3$ (J. Richter-Gebert, improving on Onn & Sturmfels [33]), but it is not clear whether the bound of 43^n can be replaced by a polynomial bound.
2. If P has a group G of symmetries, then it also has a symmetric realization. However, it is not clear whether the space of all G-symmetric realizations $\mathcal{R}^{\mathrm{G}}(P)$ is still homeomorphic to some $\mathbb{R}^k$. (It does not contain rational points in general, e.g. for the icosahedron!)

14.1.6 FOUR-DIMENSIONAL POLYTOPES AND SCHLEGEL DIAGRAMS

GLOSSARY

Schlegel diagram: A $(d-1)$-dimensional representation $\mathcal{D}(P, F)$ of a d-dimensional polytope P, obtained as follows. Take a point of view very close to (an interior point of) the facet F, and let $\mathcal{D}(P, F)$ be the decomposition of F given by all the other facets of P, as seen from this point of view.

$(d-1)$-Diagram: A polytopal decomposition $\mathcal{D}$ of a $(d-1)$-polytope F such that (1) $\mathcal{D}$ is a polytopal complex (i.e., a finite collection of polytopes closed under taking faces, such that any intersection of two polytopes in the complex is a face of each), and
(2) the intersection of any polytope in $\mathcal{D}$ with the boundary of F is a face of F (which may be empty).

Basic primary semialgebraic set defined over $\mathbb{Z}$: The solution set $S \subseteq \mathbb{R}^k$ of a finite set of equations and strict inequalities of the form $f_i(x) = 0$ resp. $g_j(x) > 0$, where the f_i and g_j are polynomials in k variables with integer coefficients.

Stable equivalence: Equivalence relation between semialgebraic sets generated by rational changes of coordinates and certain types of "stable" projections with contractible fibers. (See Richter-Gebert [26, Sect. 2.5].)
In particular, if two sets are stably equivalent, then they have the same homotopy type, and they have the same arithmetic properties with respect to subfields of $\mathbb{R}$; e.g., either both or neither of them contain a rational point.

The situation for 4-polytopes is fundamentally different from that for 3-dimensional polytopes. One reason is that there is no similar reduction of 4-polytope theory to a combinatorial (graph) problem.

The main results about graphs of d-polytopes are that they are d-connected (Balinski), and that each contains a subdivision of the complete graph on $d+1$

vertices, $K_{d+1} = G(T_d)$ (Grünbaum). In particular, all graphs of 4-polytopes are 4-connected, and none of them is planar. (See also Chapter 18.)

Schlegel diagrams provide a reasonably efficient tool for visualization of 4-polytopes: we have a fighting chance to understand some of their theory in terms of the 3-dimensional (!) geometry of Schlegel diagrams.

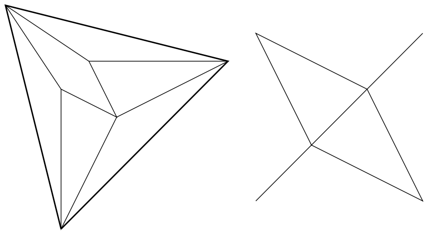


Figure 14.8
Two Schlegel diagrams of our unnamed 3-polytope, the first based on a triangle facet, the second on the "bottom square."

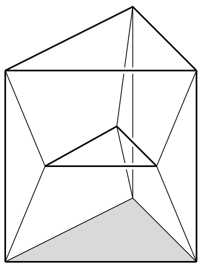


Figure 14.9
A Schlegel diagram of the product of two triangles. (This is a 4-dimensional polytope with 6 triangular prisms as facets, any two of them adjacent!)

A $(d-1)$-diagram is a polytopal complex that "looks like" a Schlegel diagram, although there are diagrams (even 2-diagrams) that are not Schlegel diagrams. The situation is somewhat nicer for *simple* 4-polytopes. These are determined by their graphs (Kalai), and they can be understood in terms of 3-diagrams: all simple 3-diagrams are projections of genuine 4-dimensional polytopes (Whiteley).

The fundamental difference between the theories for polytopes in dimensions 3 and 4 is most apparent in the contrast between Steinitz' Theorem and the following (very recent) result, which states simply that all the "nice" properties of 3-polytopes established in Steinitz' Theorem fail dramatically for 4-dimensional polytopes.

THEOREM (Richter-Gebert's Universality Theorem for 4-Polytopes [26])
The realization space of a 4-dimensional polytope can be "arbitrarily wild": for every basic primary semialgebraic set S defined over $\mathbb{Z}$ there is a 4-dimensional polytope $P[S]$ whose realization space $\mathcal{R}(P[S])$ is stably equivalent to S.
In particular, this implies the following.

- *The isotopy property fails for 4-dimensional polytopes.*
- *There are non-rational 4-polytopes: combinatorial types that cannot be realized with rational vertex coordinates.*
- *The coordinates needed to represent all combinatorial types of rational 4-polytopes with integer vertices grow doubly-exponentially with $f_0(P)$.*

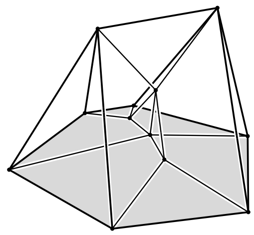


Figure 14.10
Schlegel diagram of a 4-dimensional polytope with 8 facets and 12 vertices, for which the shape of the base hexagon cannot be prescribed arbitrarily.

The complete proof of this Universality Theorem is given in [26]. One key component of the proof corresponds to another failure of a 3-dimensional phenomenon in dimension 4: for any facet (2-face) F of a 3-dimensional polytope P, the shape of F can be arbitrarily prescribed; in other words, the canonical map of realization spaces $\mathcal{R}(P) \longrightarrow \mathcal{R}(F)$ is always surjective. Richter-Gebert shows that a similar statement fails in dimension 4, even if F is a 2-dimensional pentagonal face: see Figure 14.10 for the case of a hexagon.

A problem that is left open is the structure of the realization spaces of simplicial 4-polytopes. All that is available now is a Universality Theorem for simplicial polytopes without a dimension bound (see Section 7.3.4), and a single example of a simplicial 4-polytope that violates the isotopy property, by Bokowski, Ewald & Kleinschmidt [9].

14.1.7 POLYTOPES WITH FEW VERTICES — GALE DIAGRAMS

GLOSSARY

Polytope with few vertices: A polytopes that has only a few more vertices than its dimension; usually a d-polytope with at most $d{+}4$ vertices.

(Affine) Gale diagram: A configuration of n (positive and negative) points in affine space $\mathbb{R}^{n-d-2}$ that encodes a d-polytope with n vertices uniquely up to projective transformations.

The computation of a Gale diagram is quite simple linear algebra. For this, let $V \in \mathbb{R}^{d\times n}$ be a matrix whose columns consist of coordinates for the vertices of a d-polytope. For simplicity, we assume that P is not a pyramid, and that the vertices $\{v^1, \dots, v^{d+1}\}$ affinely span $\mathbb{R}^d$. Let $\widetilde{V} \in \mathbb{R}^{(d+1)\times n}$ be obtained from V by adding an extra (terminal) row of ones. The vector configuration given by the columns of $\widetilde{V}$ represents the *oriented matroid* of P; see Chapter 7.

Now perform row operations on the matrix $\widetilde{V}$ to get it into the form $\widetilde{V} \sim (I_{d+1}|A)$, where I_{d+1} denotes a unit matrix, and $A \in \mathbb{R}^{(d+1)\times(n-d-1)}$ is a real matrix. (The row operations do not change the oriented matroid.) The columns of the matrix $\widetilde{V}^* := (-A^T|I_{n-d-1}) \in \mathbb{R}^{(n-d-1)\times n}$ then represent the dual oriented matroid. We find a vector $a \in \mathbb{R}^{n-d-1}$ that has non-zero scalar product with all the columns of $\widetilde{V}^*$, divide each column w^* of $\widetilde{V}^*$ by the value $\langle a, w\rangle$, and delete from the resulting matrix any row that affinely depends on the others, thus obtaining a matrix $W \in \mathbb{R}^{(n-d-2)\times n}$. The columns of W give a colored point configuration in $\mathbb{R}^{n-d-2}$, where *black* points are used for the columns where $\langle a, w\rangle > 0$, and *white* points for the others. This colored point configuration represents an affine Gale diagram of P.

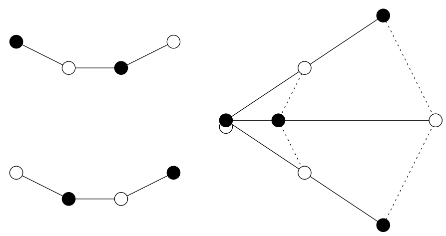


Figure 14.11
Two affine Gale diagrams of 4-dimensional polytopes: for a non-cyclic neighborly polytope with 8 vertices, and for the polar (with 8 vertices) of the polytope with 8 facets from Figure 14.10, for which the shape of a hexagon face cannot be prescibed arbitrarily.

It turns out that an affine configuration of colored points (consisting of n points that affinely span $\mathbb{R}^e$) represents a polytope (with n vertices, of dimension $n-e-2$) if and only if the following criterion is met: For any hyperplane spanned by some of the points, and for each side of it, the number of black points on this side, plus the number of white points on the other side, is at least 2.

The final information one needs is how to read off properties of a polytope from its affine Gale diagram. Here the criterion is that a set of points represents a face if and only if the following condition is satisfied: the colored points *not* in the set support an affine dependency, with positive coefficients on the black points, and with negative coefficients on the white points. Equivalently, the convex hull of all the black points not in our set, and the convex hull of all the white points not in the set, intersect in their relative interiors.

Affine Gale diagrams have been *very* successfully used to study and classify polytopes with few vertices.

d+1 vertices: The only d-polytopes with $d+1$ vertices are the d-simplices.

d+2 vertices: There are exactly $\lfloor d^2/4\rfloor$ combinatorial types of d-polytopes with $d+2$ vertices; among these, $\lfloor d/2\rfloor$ types are simplicial. This corresponds to the situation of 0-dimensional affine Gale diagrams.

d+3 vertices: All d-polytopes with $d+3$ vertices are realizable with (small) integral coordinates and satisfy the isotopy property: all this can be easily analyzed in terms of 1-dimensional affine Gale diagrams.

d+4 vertices: Here anything can go wrong: the universality theorem for oriented matroids of rank 3 yields a universality theorem for simplicial d-polytopes with $d+4$ vertices. (See Section 7.3.4.)

We refer to [34, Lect. 6] for a detailed introduction to affine Gale diagrams.

14.2 METRIC PROPERTIES

The combinatorial data of a polytope — vertices, edges,..., facets — have their counterparts in genuine geometric data, such as face volumes, surface areas, quermassintegrals and the like. In this second half of the chapter, we give a brief sketch of some key geometric concepts related to polytopes.

However, the topics of combinatorial and of geometric invariants are not disjoint at all: much of the beauty of the theory stems from the subtle interplay between the two sides. Thus, the computation of volumes inevitably leads to the construction of triangulations (explicitly or implicitly), mixed volumes lead to mixed subdivisions of Minkowski sums (one "hot topic" for current research in the area), quermassintegrals relate to face enumeration, and so on.

Furthermore, the study of polytopes yields a powerful approach to the theory of convex bodies: sometimes one can extend properties of polytopes to arbitrary convex bodies by approximation [30]. (However, there are also properties valid for polytopes that fail for convex bodies in general. This bug/feature is designed to keep the game interesting.)

14.2.1 VOLUME AND SURFACE AREA

GLOSSARY

Volume of a d-simplex T: $V(T) = \frac{1}{d!}\left|\det\begin{pmatrix} v^0 & \cdots & v^d \\ 1 & \cdots & 1 \end{pmatrix}\right|$ for $T = \mathrm{conv}\{v^0,\dots,v^d\}$, with $v^0,\dots,v^d \in \mathbb{R}^d$.

Subdivision of a polytope P: A collection of polytopes $P_1,\dots,P_l \subseteq \mathbb{R}^d$ such that $P = \bigcup P_i$, and for $i \neq j$ we have that $P_i \cap P_j$ is a proper face of P_i and P_j (possibly empty). In this case we write $P = \uplus P_i$.

Triangulation of a polytope: A subdivision into simplices. (See Chapter 15.)

Volume of a d-polytope: $\sum_{T\in\Delta(P)} V(T)$, where $\Delta(P)$ is a triangulation of P.

k-Volume $V^k(P)$ of a k-polytope $P \subseteq \mathbb{R}^d$: The volume of P, computed with respect to the k-dimensional Euclidean measure induced on aff(P).

Surface area of a d-polytope P: $\sum_{T\in\Delta(P),\,F\in\mathcal{F}_{d-1}(P)} V^{d-1}(T\cap F)$, where $\Delta(P)$ is a triangulation of P.

The volume $V(P)$ (i.e., the d-dimensional Lebesgue measure) and the surface area $F(P)$ of a d-polytope $P \subseteq \mathbb{R}^d$ can be derived from any triangulation of P, since volumes of simplices are easy to compute. The crux for this is in the (efficient?) generation of a triangulation, a topic on which Chapters 15 and 23 of this Handbook have more to say.

The following recursive approach only implicitly generates a triangulation, but derives explicit volume formulas. Let $P \subseteq \mathbb{R}^d$ ($P \neq \emptyset$) be a polytope. If $d = 0$ then we set $V(P) = 1$. Otherwise we set $\mathcal{S}_{d-1}(P) := \{u \in S^{d-1} : \dim(H(P,u) \cap P) =$

$d-1\}$, and use this to define the volume of P as

$$V(P) \quad := \quad \frac{1}{d} \sum_{u \in \mathcal{S}_{d-1}(P)} h(P,u) \cdot V^{d-1}(H(P,u) \cap P).$$

Thus, for any d-polytope the volume is a sum of its facet volumes, each weighted by $1/d$ times its signed distance from the origin. Geometrically, this can be interpreted as follows. Assume for simplicity that the origin is in the interior of P. Then the collection $\{\mathrm{conv}(F \cup \{0\}) : F \in \mathcal{F}_{d-1}(P)\}$ is a subdivision of P into d-dimensional pyramids, where the base of $\mathrm{conv}(F \cup \{0\})$ has $(d-1)$-dimensional volume $V^{d-1}(F)$ — to be computed recursively —, the height of the pyramid is $h(P, u^F)$, and thus its volume is $\frac{1}{d} h(P, u^F) \cdot V^{d-1}(F)$; compare to Figure 14.12. (The formula remains valid even if the origin is outside P or on its boundary.)

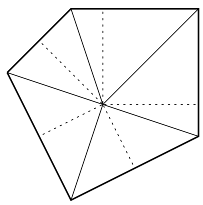


Figure 14.12
This pentagon, with the origin in its interior, is decomposed into five pyramids (triangles), each with one of the pentagon facets (edges) F_i as its base. For each pyramid, the height, of length $h(P, u^{F_i})$, is drawn as a dotted line.

Note that $V(P) \geq 0$. This holds with strict inequality if and only if the polytope P has full dimension d. The surface area $F(P)$ can also be expressed as

$$F(P) \quad = \quad \sum_{u \in \mathcal{S}_{d-1}(P)} V^{d-1}(H(P,u) \cap P).$$

Thus for a d-polytope the surface area is the sum of the $(d-1)$-volumes of its facets. If $\dim(P) = d-1$, then $F(P)$ is twice the $(d-1)$-volume of P. One has $F(P) = 0$ if and only if $\dim(P) < d-1$.

Both the volume and the surface area are continuous, monotone and invariant with respect to rigid motions. $V(\cdot)$ is homogeneous of degree d, i.e., $V(\mu P) = \mu^d V(P)$ for $\mu \geq 0$, and $F(\cdot)$ is homogeneous of degree $d-1$. For further properties of the functionals $V(\cdot)$ and $F(\cdot)$ see [17] and [11].

The following table gives the numbers of k-faces, the volume and surface area of the d-cube C_d (with edge length 2), of the crosspolytope C_d^{Δ} with edge length $\sqrt{2}$, and of the regular simplex T_d with edge length $\sqrt{2}$.

Polytope	$f_k(\cdot)$	Volume	Surface area
C_d	$2^{d-k}\binom{d}{k}$	2^d	$2d \cdot 2^{d-1}$
C_d^{Δ}	$2^{k+1}\binom{d}{k+1}$	$\frac{2^d}{d!}$	$2^d \frac{\sqrt{d}}{(d-1)!}$
T_d	$\binom{d+1}{k+1}$	$\frac{\sqrt{d+1}}{d!}$	$(d+1) \cdot \frac{\sqrt{d}}{(d-1)!}$

14.2.2 MIXED VOLUMES

GLOSSARY

Volume polynomial: The volume of the Minkowski sum $\lambda_1 P_1 + \lambda_2 P_2 + \ldots + \lambda_r P_r$, which is a homogeneous polynomial in $\lambda_1, \ldots, \lambda_r$. (Here the P_i may be convex polytopes of any dimension, or more general (closed, bounded) convex sets.)

Mixed volumes: The coefficients of the volume polynomial of $P_1, \ldots, P_r$.

Normal cone: The normal cone $N(F, P)$ of a face is the set of all vectors $v \in \mathbb{R}^d$ such that the supporting hyperplane $H(P, v)$ contains F, i.e.,

$$N(F, P) = \left\{v \in \mathbb{R}^d : F \subseteq H(P, v) \cap P\right\}.$$

THEOREM (Mixed volumes) *Let $P_1, \ldots, P_r \subseteq \mathbb{R}^d$ be polytopes, $r \geq 1$, and $\lambda_1, \ldots, \lambda_r \geq 0$. The volume of $\lambda_1 P_1 + \ldots + \lambda_r P_r$ is a homogeneous polynomial in $\lambda_1, \ldots, \lambda_r$ of degree d. Thus it can be written in the form*

$$V(\lambda_1 P_1 + \ldots + \lambda_r P_r) = \sum_{(i(1),\ldots,i(d)) \in \{1,2,\ldots,r\}^d} \lambda_{i(1)} \cdots \lambda_{i(d)} \cdot V(P_{i(1)}, \ldots, P_{i(d)}).$$

The coefficients in this expansion are symmetric in their indices. Furthermore, the coefficient $V(P_{i(1)}, \ldots, P_{i(d)})$ depends only on $P_{i(1)}, \ldots, P_{i(d)}$. It is called the mixed volume *of the polytopes $P_{i(1)}, \ldots, P_{i(d)}$.*

With the abbreviation

$$V(P_1, k_1; \ldots; P_r, k_r) \quad := \quad V(\underbrace{P_1, \ldots, P_1}_{k_1 \text{ times}}, \ldots, \underbrace{P_r, \ldots, P_r}_{k_r \text{ times}}),$$

the polynomial becomes

$$V(\lambda_1 P_1 + \ldots + \lambda_r P_r) \quad = \sum_{\substack{k_1,\ldots,k_r \geq 0 \\ k_1+\ldots+k_r = d}} \binom{d}{k_1, \ldots, k_r} \lambda_1^{k_1} \cdots \lambda_r^{k_r} V(P_1, k_1; \ldots; P_r, k_r).$$

In particular, the volume of the polytope P_i is given by the mixed volume $V(P_1, 0; \ldots; P_i, d; \ldots; P_r, 0)$. The theorem is also valid for arbitrary convex bodies: a good example where the general case can be derived from the polytope case by approximation. For more about the properties of mixed volumes from different points of view see Schneider [30], Sangwine-Yager [28] and McMullen [24].

The definition of the mixed volumes as coefficients of a polynomial is somewhat unsatisfactory. Only recently, Schneider [31] gave the following explicit rule, which generalizes an earlier result of Betke [4] for the case $r = 2$. It uses information about the normal cones at certain faces. For this note that $N(F, P)$ is a finitely generated cone, which can be written explicitly as the sum of the orthogonal complement of aff(P) and the positive hull of those unit vectors u that are both parallel to aff(P), and induce supporting hyperplanes $H(P, u)$ that contain a facet of P including F. Thus, for $P \subseteq \mathbb{R}^d$ the dimension of $N(F, P)$ is $d - \dim(F)$.

THEOREM (Schneider's summation formula) *Let $P_1,\ldots,P_r \subseteq \mathbb{R}^d$ be polytopes, $r \geq 2$. Let $x^1,\ldots,x^r \in \mathbb{R}^d$ such that $x^1+\ldots+x^r = 0$, $(x^1,\ldots,x^r) \neq (0,\ldots,0)$, and*

$$\bigcap_{i=1}^{r} \left(\mathrm{relint} N(F_i,P_i) - x^i\right) \quad = \quad \emptyset,$$

whenever F_i is a face of P_i and $\dim(F_1)+\ldots+\dim(F_r) > d$. Then

$$\binom{d}{k_1,\ldots,k_r} V(P_1,k_1;\ldots;P_r,k_r) = \sum_{(F_1,\ldots,F_r)} V(F_1+\ldots+F_r),$$

where the summation extends over the r-tuples $(F_1,\ldots,F_r)$ of k_i-faces F_i of P_i with $\dim(F_1+\ldots+F_r) = d$ and $\bigcap_{i=1}^r \left(N(F_i,P_i) - x^i\right) \neq \emptyset$.

The choice of the vectors $x^1,\ldots,x^r$ implies that the selected k_i-faces $F_i \subseteq P_i$ of a summand $F_1+\ldots+F_r$ are contained in complementary subspaces. Hence one may also write

$$\binom{d}{k_1,\ldots,k_r} V(P_1,k_1;\ \ldots\ ;P_r,k_r) = \sum_{(F_1,\ldots,F_r)} [F_1,\ldots,F_r]\cdot V^{k_1}(F_1)\cdots V^{k_r}(F_r),$$

where $[F_1,\ldots,F_r]$ denotes the volume of the parallelepiped that is the sum of unit cubes in the affine hulls of $F_1,\ldots,F_r$.

Finally, we remark that the selected sums of faces in the formula of the theorem form a subdivision of the polytope $P_1+\ldots+P_r$, i.e.,

$$P_1+\ldots+P_r \quad = \quad \biguplus_{(F_1,\ldots,F_r)} (F_1+\ldots+F_r)\,.$$

See Figure 14.13 for an example.

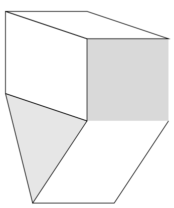


Figure 14.13
Here the Minkowski sum of a square P_1 and a triangle P_2 is decomposed into translates of P_1 and of P_2 (this corresponds to two summands with $F_1 = P_1$ resp. $F_2 = P_2$), together with three "mixed" faces that arise as sums F_1+F_2, where F_1 and F_2 are faces of P_1 and P_2 (corresponding to summands with $\dim(F_1) = \dim(F_2) = 1$).

Volumes of Zonotopes

If all summands in a Minkowski sum $Z = P_1+\ldots+P_r$ are line segments, say $P_i = p^i + [0,1]z^i = \mathrm{conv}\{p^i, p^i+z^i\}$ with $p^i, z^i \in \mathbb{R}^d$ for $1 \leq i \leq r$, then the resulting polytope Z is a zonotope. In this case the summation rule immediately gives $V(P_1,k_1;\ldots;P_r,k_r) = 0$ if the vectors

$$\underbrace{z^1,\ldots,z^1}_{k_1 \text{ times}},\ \ldots,\ \underbrace{z^r,\ldots,z^r}_{k_r \text{ times}}$$

are linearly dependent. (This can also be seen directly from dimension considerations.) Otherwise, for $k_{i(1)} = k_{i(2)} = \ldots = k_{i(d)} = 1$, say,

$$V(P_1, k_1; \ldots; P_r, k_r) = \frac{1}{d!} \left| \det \left(z^{i(1)}, z^{i(2)}, \ldots, z^{i(d)} \right) \right| .$$

Therefore, one obtains McMullen's formula for the volume of the zonotope Z:

$$V(Z) = \sum_{1 \le i(1) < i(2) < \cdots < i(d) \le r} \left| \det(z^{i(1)}, \ldots, z^{i(d)}) \right| .$$

14.2.3 QUERMASSINTEGRALS AND INTRINSIC VOLUMES

GLOSSARY

i-th Quermassintegral $\boldsymbol{W_i(P)}$**:** The mixed volume $V(P, d-i; B_d, i)$ of a polytope P and the d-dimensional unit ball B_d.

$\boldsymbol{\kappa_d}$**:** The volume (Lebesgue measure) of B_d. (Hence $\kappa_0 = 1$, $\kappa_1 = 2$, $\kappa_2 = \pi$, etc.)

i-th Intrinsic volume $\boldsymbol{V_i(P)}$**:** The $(d-i)$-th quermassintegral, scaled by the constant $\binom{d}{i}/\kappa_{d-i}$.

Outer parallel body of P at distance $\boldsymbol{\lambda}$**:** The convex body $P + \lambda B_d$ for some $\lambda > 0$.

External angle $\boldsymbol{\gamma(F,P)}$**:** The volume of $\left(\mathrm{lin}(F - x^F) + N(F,P)\right) \cap B_d$ divided by κ_d, for $x^F \in \mathrm{relint}(F)$. Thus $\gamma(F,P)$ is the "fraction of $\mathbb{R}^d$ taken up by $\mathrm{lin}(F - x^F) + N(F,P)$."
Equivalently, the external angle at a k-face F is the fraction of the spherical volume of S covered by $N(F,P) \cap S$, where S denotes the $(d-k-1)$-dimensional unit sphere in $\mathrm{lin}(N(F,P))$.

Internal angle $\boldsymbol{\beta(F,G)}$ ***for faces*** $\boldsymbol{F \subseteq G}$**:** The "fraction" of the space $\mathrm{lin}\{G - x^F\}$ taken up by the cone $\mathrm{pos}\{x - x^F : x \in G\}$, for $x^F \in \mathrm{relint}(F)$.
(A detailed discussion of relations between external and internal angles can be found in McMullen [23].)

The quermassintegrals are generalizations of both the volume and the surface area of P. In fact, they can also be seen as the continuous convex geometry analogs of face numbers.

For a polytope $P \subseteq \mathbb{R}^d$ and the d-dimensional unit ball B_d, the mixed volume formula, applied to the outer parallel body $P + \lambda B_d$, gives

$$V(P + \lambda B_d) = \sum_{i=0}^{d} \binom{d}{i} \lambda^i W_i(P)$$

with the convention $W_i(P) = V(P, d-i; B_d, i)$. This formula is known as the Steiner polynomial. The mixed volume $W_i(P)$, the i-th quermassintegral of P, is an important quantity and of significant geometric interest [17] [30]. As special cases, $W_0(P) = V(P)$ is the volume, $dW_1(P) = F(P)$ is the surface area, and $W_d(P) = \kappa_d$.

For the geometric interpretation of $W_i(P)$ for polytopes, we use a normalization of the quermassintegrals due to McMullen [23]: For $0 \le i \le d$, the i-th intrinsic volume of P is defined by

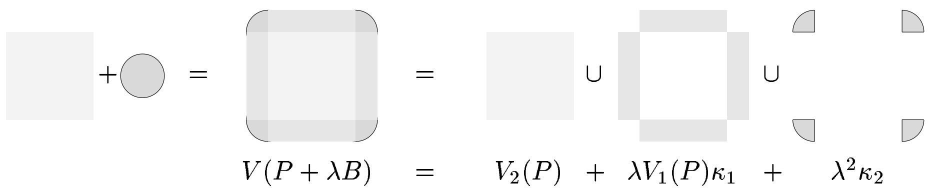
$$V_i(P) \quad := \quad \frac{\binom{d}{i}}{\kappa_{d-i}} W_{d-i}(P).$$

With this notation the Steiner polynomial can be written as

$$V(P + \lambda B_d) = \sum_{i=0}^{d} \lambda^{d-i} \kappa_{d-i} V_i(P).$$

(See Figure 14.14 for an example.) $V_d(P)$ is the volume of P, $V_{d-1}(P)$ is half the surface area and $V_0(P) = 1$. One advantage of this normalization is that the intrinsic volumes are unchanged if P is embedded in some Euclidean space of different dimension. Thus, for $\dim(P) = k \le d$, $V_k(P)$ is the ordinary k-volume of P with respect to the Euclidean structure induced in aff(P).

Figure 14.14
The Minkowski sum of a square P with a ball λB^2 yields the outer parallel body. This outer parallel body can be decomposed into pieces, whose volumes, $V(P)$, $\lambda V_1(P)\kappa_1$ and $\lambda^2\kappa_2$ correspond to the three terms in the Steiner polynomial.



For a $(\dim(P) - 2)$-face F the concept of external angle (see the glossary) reduces to the "usual" concept: then the external angle is given by $\frac{1}{2\pi} \arccos\langle u^{F_1}, u^{F_2}\rangle$ for unit normal vectors $u^{F_1}, u^{F_2} \in S^{d-1}$ to the facets F_1, F_2 with $F_1 \cap F_2 = F$. (One has $\gamma(P, P) = 1$ for the polytope itself and $\gamma(F, P) = 1/2$ for each facet F.) Using this concept, we get

$$V_k(P) = \sum_{F \in \mathcal{F}_k(P)} \gamma(F, P) \cdot V^k(F).$$

Some Computations

In principle, one can use the external angle formula to determine the intrinsic volumes of a given polytope, but in general it is hard to calculate external angles. Indeed, for the computation of spherical volumes there are explicit formulas only in small dimensions.

In what follows, we give formulas for the intrinsic volumes of the polytopes C_d, C_d^Δ and T_d. For this we identify the k-faces of C_d with the k-cube C_k and the k-faces of C_d^Δ and of T_d with T_k, for $0 \le k < d$.

The case of the cube C_d is rather trivial. Since $\gamma(C_k, C_d) = 2^{-(d-k)}$ one gets (see the table in Sect. 14.2.1)

$$V_k(C_d) = 2^k \binom{d}{k}.$$

For the regular simplex T_d we have

$$V_k(T_d) = \binom{d+1}{k+1} \cdot \frac{\sqrt{k+1}}{k!} \cdot \gamma(T_k, T_d).$$

An explicit formula for the external angles of a regular simplex by Ruben (see [18]) is:

$$\gamma(T_k, T_d) = \sqrt{\frac{k+1}{\pi}} \int_{-\infty}^{\infty} e^{-(k+1)x^2} \left(\frac{1}{\sqrt{\pi}} \int_{-\infty}^{x} e^{-y^2} dy \right)^{d-k} dx.$$

For the regular crosspolytope we find for $k \leq d-1$ that

$$V_k(C_d^\Delta) = 2^{k+1} \binom{d}{k+1} \cdot \frac{\sqrt{k+1}}{k!} \cdot \gamma(T_k, C_d^\Delta).$$

For this, the external angles of C_d^Δ were determined by Betke & Henk [5]:

$$\gamma(T_k, C_d^\Delta) = \sqrt{\frac{k+1}{\pi}} \int_{0}^{\infty} e^{-(k+1)x^2} \left(\frac{2}{\sqrt{\pi}} \int_{0}^{x} e^{-y^2} dy \right)^{d-k-1} dx.$$

An Application

External angles and internal angles play a crucial role in work by Affentranger & Schneider [1] (see also [2]), who computed the expected number of k-faces of the orthogonal projection of a polytope $P \subseteq \mathbb{R}^d$ onto a randomly chosen isotropic subspace of dimension n. Let $E[f_k(P;n)]$ be that number. Then for $0 \leq k < n \leq d-1$ it was shown that

$$E[f_k(P;n)] \quad = \quad 2 \sum_{m \geq 0} \sum_{F \in \mathcal{F}_k(P)} \sum_{\substack{G \in \mathcal{F}_{n-1-2m}(P) \\ F \subseteq G}} \beta(F,G)\gamma(G,P),$$

where $\beta(F,G)$ is the internal angle of the face F with respect to a face $G \supseteq F$.

In the sequel we apply the above formula to the polytopes C_d, C_d^Δ and T_d. For the cubes one has $\beta(C_k, C_l) = (1/2)^{l-k}$, while the number of l-faces of C_d containing any given k-face is equal to $\binom{d-k}{l-k}$. Hence

$$E[f_k(C_d;n)] = 2\binom{d}{k} \sum_{m \geq 0} \binom{d-k}{n-1-k-2m}.$$

In particular, $E[f_k(C_d;d-1)] = (2^{d-k}-2)\binom{d}{k}$.

For the crosspolytope C_d^Δ the number of l-faces which contain a k-face is equal to $2^{l-k}\binom{d-k-1}{l-k}$. Thus

$$E[f_k(C_d^\Delta;n)] = \\ 2\binom{d}{k+1}\sum_{m\geq 0} 2^{n-2m}\binom{d-k-1}{n-1-k-2m}\beta(T_k,T_{n-1-2m})\gamma(T_{n-1-2m},C_d^\Delta).$$

In the same way one obtains for T_d

$$E[f_k(T_d;n)] = \\ 2\binom{d+1}{k+1}\sum_{m\geq 0}\binom{d-k}{n-1-k-2m}\beta(T_k,T_{n-1-2m})\gamma(T_{n-1-2m},T_d).$$

For the last two formulas one needs the internal angles $\beta(T_k,T_l)$ of the regular simplex T_d, for $0 \leq k \leq l \leq d$. For this, one has the following complex integral [8]:

$$\beta(T_k,T_l) = \frac{(k+1+l)^{1/2}(k+1)^{(l-1)/2}}{\pi^{(l+1)/2}}\int_{-\infty}^{\infty} e^{-w^2}\left(\int_0^{\infty} e^{-(k+1)y^2+2iwy}dy\right)^l dw.$$

Using this formula one can determine the asymptotic behavior of $E[f_k(C_d^\Delta;n)]$ and $E[f_k(T_d;n)]$ as n tends to infinity [8].

FURTHER READING

The classic account of the combinatorial theory of convex polytopes was given by Grünbaum in 1967 [16]. It inspired and guided a great part of the subsequent research in the field. Besides the subsequent chapters of this handbook, we refer to the recent handbook surveys by Klee & Kleinschmidt [22] and by Bayer & Lee [3] for further reading.

For the geometric theory of convex bodies, and especially convex polytopes, a classic is Bonnesen & Fenchel [11]. Here we refer to the Handbook of Convex Geometry [15] for recent surveys, and to Schneider [30] for an excellent recent monograph. As for the algorithmic aspects of computing volumes etc., we refer to Chapter 29 of this handbook, on Computational Convexity, and to the additional references given there.

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