Rotations, Translations and Symmetry Detection for Complexified Curves

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Abstract. A plane algebraic curve can be represented as the zero-set of a polynomial in two - or if one takes homogenous coordinates: three - variables. The coefficients of the polynomial determine the curve uniquely. Thus features of the curve, like for instance rotation symmetry, must find their correspondence in the algebraic structure of the coefficients of the polynomial. In this article we will investigate how one can extract geometric curve features from the algebraic description of the curve. In particular, we will study a certain complex representation of the polynomial, which is very appropriate for the task of feature detection. In this complex representation actions on the curve parameters induced by geometric rotations or translations of the plane become very simple. Invariant expressions in the complexified parameters and also normal forms are easily accessible. Furthermore our representation allows the detection of rotation symmetry simply by looking at the indices of all non-vanishing complex parameters.

1 Introduction

A plane algebraic curve $\mathcal{C}_f$ is a zero set of a non-constant polynomial $f(x,y)$. A point $p=(x_p,y_p)$ is on $\mathcal{C}_f$ if $f(x_p,y_p)=0$. We will, as common in the literature, prefer a homogenized setup in which a third homogenizing variable h shows up. A point (x,y) is represented by its homogeneous counterpart $(x,y,1)$. Vectors that differ only by a common multiple are identified. By this we gain a cleaner algebraic setup and we also properly include points at infinity that have the form $(x,y,0)$. In this setup $f(x,y)$ becomes $f(x,y,h)$. Let $g(x,y,h)$ be a square-free factorization of $f(x,y,z)$. Let the degree of g be d. Then $g(x,y,h)$ also defines the degree d of the curve $\mathcal{C}_f$: $d=\deg(\mathcal{C}_f)=\deg(g(x,y,h))$. Thus a curve $\mathcal{C}_f$ of degree d can be described by

$$\left\{ (x,y,h) \in \mathbb{RP}^2 \;\middle|\; f(x,y,h) = \sum_{k+l\leq d} A_{kl}\; x^k\; y^l\; h^{d-k-l} = 0 \right\} \quad . \tag{1}$$

While the polynomial $f(x,y,h)$ uniquely determines the corresponding curve $\mathcal{C}_f$, the opposite is not true, even if the degree of $f(x,y,h)$ is chosen to be minimal. With an arbitrary $\lambda \neq 0$ the polynomial $\lambda \cdot f(x,y,h) = 0$ defines the same curve as $f(x,y,h)$. Nevertheless, we will often say that a homogeneous non-constant polynomial $f(x,y,h)$ *is* a curve $\mathcal{C}_f$ because it fully determines $\mathcal{C}_f$.

This paper is in a sense the second step in an ongoing project that heads towards automatic classification of algebraic curves that are given by a set of sample points with high arithmetic precision. Such curves, given by sample points, arise for instance as *loci* in dynamic geometry systems. While [2] dealt with the problem of extracting an algebraic equation $f(x,y,h) = 0$ from a set of sample points on $\mathcal{C}_f$, this article deals with the extraction of *features* and *normal forms* from the coefficients of $f(x,y,h)$.

Studying normal forms of curves is tied to studying a transformation group and the representation of a curve. In equation (1), the curve $\mathcal{C}_f$ was given in monomial form with A_{kl} $(k+l \leq d)$ as curve coefficients. There are of course infinitely many other ways to represent a curve. They may for instance arise by simple coordinate transformations. In Section 2.2 we will substitute x by $z = x+iy$ and y by $\overline{z} = x - iy$. Our polynomial then becomes $F(z,\overline{z},h) = \sum_{k+d\leq d} C_{kl} z^k \overline{z}^l h^{d-k-l}$. The coefficients C_{kl} of F arise by applying a certain transformation matrix to the coefficients A_{kl} of f. In this new coordinate system rotations around the origin become very simple operations. Translations are slightly more complicated than rotations but still easy to handle.

Extracting invariant expressions for curves in this representation is a simple task. In case of rotations around the origin, these expressions can be extracted only by looking at the indices k and l of all non-vanishing coefficients C_{kl} of F. Also a normal form is easily derived. A normal form in our sense is defined by

Definition 1. *Let G be a transformation group of the plane, e.g. the group of all rotations or translations. This introduces an equivalence relation $\sim$ on the set of all algebraic curves of a fixed degree d. For two curves f and g: $f \sim g$ if and only if there is a transformation $\tau \in G$ such that τ maps f onto g. Thus $\sim$ subdivides the set of our curves into classes. A designated representation system with respect to this transformation group will be called a **normal form**. A curve is in normal form, if it is given as one of the designated representatives.*

We present normal forms with respect to rotations and translations and show how a given curve must be transformed such that it is in normal form.

In the second part of the paper we deal with translations. We will see that under translations we can easily single out coefficients that can be made zero. Taking the presented rotational and translatorial normal forms together, we will end up with a normal form for curves with respect to orientation preserving Euclidean transformations. Our procedure will apply to *almost all* curves. However, for a zero-set of parameters we will not arrive at a unique normal form. This set of exceptions has very interesting algebraic and geometric properties.

Besides being easy to calculate our normal-forms permit elegant information extraction. We show how to detect a potential rotational symmetry in Section 4. In a third part [3] of this series of papers we will deal with the detection of features under projective transformations of the curve. However this issue will not be covered here.

2 Transformations of the plane and rotations

We show how linear transformations of the plane act on curves contained in this plane. We do this by focusing ourselves to the case of rotations.

A rotation is a Euclidean motion with a fixpoint: the center of rotation. Any rotation can be performed in three steps: First apply a translation τ, which maps the center of rotation to the origin. Then perform the rotation by the desired angle φ and apply τ^{-1} afterwards. We are dealing with translation later on (see Chapter 3). Thus, when speaking of rotations, we confine ourselves here to rotations around the origin with an arbitrary rotation angle φ.

2.1 Effects of rotations on curve parameters

A rotation can be described very easily: Abbreviating $\cos(\varphi) = c$ and $\sin(\varphi) = s$, a point $p = (x, y, h)$ with homogenizing component h is rotated by applying the map

$$\begin{pmatrix} x \\ y \\ h \end{pmatrix} \longmapsto \begin{pmatrix} c & -s & 0 \\ s & c & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ h \end{pmatrix} \quad .$$

The question now is, what happens to the polynomial $f(x, y, h)$ of the curve $\mathcal{C}_f$, when the plane is rotated. As we have seen, a curve of degree d is given by a zero set of a polynomial equation:

$$f(x, y, h) = \sum_{k+l=0}^{d} A_{kl} \cdot x^k y^l h^{d-k-l} = 0 \quad . \tag{2}$$

The A_{kl} are the curve coefficients. We will now study the effect of the geometric transformation on these curve coefficients. In essence Equation (2) may be interpreted as a scalar product between two vectors:

$$f(x, y, h) = A^T X \quad , \tag{3}$$

where X is a shorthand for the vector that contains all monomials of degree d in x, y and h. The curve parameter vector A consists of the coefficients A_{kl}. For further reference we introduce a formal order on the coefficients, which are variables with two indices:

Definition 2. *Let $\Omega = \{\omega_{kl} \mid k,l \in I\}$ be a set of variables with two indices k and l originating from an index set $I \subset \mathbb{N}$. Then an order $\succ$ on Ω is defined by:*

- $\omega_{kl} \succ \omega_{pq}$, *if* $k+l > p+q$ *and*
- $\omega_{kl} \succ \omega_{pq}$, *whenever* $k+l = p+q$ *and* $k > p$.

Thus equality holds only if $k = p$ and $l = q$.
Analogously we define $\prec$.

Without loss of generality we may assume that the curve parameter vector A is arranged according to $\prec$:

$$A^T = \left(A_{00}, A_{10}, A_{01}, A_{20}, A_{11}, A_{02}, ...\right) \quad .$$

Of course A determines a curve uniquely. Consequently knowing how A behaves under rotation of the plane means knowledge of the curve's behavior.

Let us first examine the variable vector X. For the sake of simplicity let $h = 1$, $c = \cos(\varphi)$ and $s = \sin(\varphi)$. Rotating the plane around the origin the components $x^k y^l$ of X are transformed according to the transformation of the plane:

$$x^k \cdot y^l \longmapsto \widehat{x}^k \cdot \widehat{y}^l = (cx - sy)^k \cdot (sx + cy)^l \quad . \tag{4}$$

Expanding the right side, we obviously get a homogeneous polynomial in x and y of degree $k + l$. The coefficients of this polynomial are expressions in c and s. This means that the transformed expression is a linear combination in all possible products $x^p y^q$ but *only* with $p + q = k + l$. Consequently a rotation of the plane induces a linear transformation R of the variable vector X. Moreover the transformation R has block-shape and is invertible.

The connection between the transformation of X and the transformation of the curve parameters is indicated by (3). We have

$$A^T X = A^T \cdot \left(R^{-1} R\right) \cdot X = \left(R^{-T} A\right)^T \cdot RX = \widetilde{A}^T \widetilde{X} \tag{5}$$

where R^{-T} is the transposed inverse of R. Thus, when the variable vector X is transformed by *any* matrix R, the curve coefficient vector A is transformed by R^{-T}. Thereby R may be induced by a rotation of the plane, like here, or by a translation, which we will study later. The vectors X and A transform contragrediently, i.e. R counter acts on X and A.

To get to know how R^{-T} looks like, we recall that R was induced by a rotation of the plane by an angle φ. The matrix R may be extracted by writing Equation (4) simultaneously for any $\widetilde{x}^k \widetilde{y}^l$. Consequently R^{-1} is an analogous matrix induced by a rotation by $-\varphi$. With $\cos(-\varphi) = \cos(\varphi)$ and $\sin(-\varphi) = -\sin(\varphi)$ the transition from R to R^{-1} is merely a substitution of s by $-s$. Transposing

R^{-1} we get R^{-T} acting on the curve parameters:

$$\begin{aligned}&\left(A_{00}|A_{10}\ A_{01}|A_{20}\ A_{11}\ A_{02}|A_{30}\ A_{21}\ A_{12}\ A_{03}|\dots\right)^T \mapsto R^{-T}A = \\ &\underbrace{\left(\begin{array}{c|cc|ccc|cccc|c} 1 & & & & & & & & & & \\ \hline & c & -s & & & & & & & & \\ & s & c & & & & & & & & \\ \hline & & & c^2 & -cs & s^2 & & & & & \\ & & & 2cs & c^2-s^2 & -2cs & & & & & \\ & & & s^2 & cs & c^2 & & & & & \\ \hline & & & & & & c^3 & -c^2s & cs^2 & -s^3 & \\ & & & & & & 3c^2s & c^3-2cs^2 & s^3-2c^2s & 3cs^2 & \\ & & & & & & 3cs^2 & 2c^2s-s^3 & c^3-2cs^2 & -3c^2s & \\ & & & & & & s^3 & cs^2 & c^2s & c^3 & \\ \hline & & & & & & & & & & \ddots \end{array}\right)}_{=R^{-T}} \left(\begin{array}{c} A_{00} \\ \hline A_{10} \\ A_{01} \\ \hline A_{20} \\ A_{11} \\ A_{02} \\ \hline A_{30} \\ A_{21} \\ A_{12} \\ A_{03} \\ \hline \vdots \end{array}\right)\end{aligned} \tag{6}$$

For example by using Equation (4) and (5), we get

$$\begin{aligned}\widehat{x}^2 &= (cx-sy)^2 = c^2x^2 - 2csxy + s^2y^2 \\ \widehat{x}\widetilde{y} &= (cx-sy)(sx+cy) = csx^2 + (c^2-s^2)xy - csy^2 \\ \widehat{y}^2 &= (sx+cy)^2 = s^2x^2 + 2csxy + c^2y^2\end{aligned}$$

And thus

$$\begin{pmatrix}\widehat{x}^2 \\ \widehat{x}\widehat{y} \\ \widehat{y}^2\end{pmatrix} = \begin{pmatrix} c^2 & -2cs & s^2 \\ cs & c^2-s^2 & -cs \\ s^2 & 2cs & c^2 \end{pmatrix} \begin{pmatrix} x^2 \\ xy \\ y^2 \end{pmatrix}$$

The three by three matrix is a block of R. Inverting and transposing this block we obtain the corresponding block of R^{-T} shown in (6).

Despite exhibiting rich structure, this matrix is not very practical for further examinations. The curve describing polynomial was obviously given in a representation not suitable for easy displaying the effects of a rotation on curve coefficients.

2.2 Complexification

In what follows, we will describe and use a complex representation for curves, which suits better. There an analogous version of Equation (5) holds, showing that a transformation of the curve can be viewed as transformation of the parameter vector. What makes this representation better is the simpler transformation of the parameter vector under transformations of the plane.

We will now change the representation of curves. They will still be zero sets of polynomials but of complex polynomials. We change the representation by (complex) linear transforming our plane:

$$\begin{pmatrix} x \\ y \\ h \end{pmatrix} \longmapsto V \cdot \begin{pmatrix} x \\ y \\ h \end{pmatrix} = \begin{pmatrix} 1 & i & 0 \\ 1 & -i & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ h \end{pmatrix} = \begin{pmatrix} x+iy \\ x-iy \\ h \end{pmatrix} = \begin{pmatrix} z \\ \overline{z} \\ h \end{pmatrix} \quad . \tag{7}$$

A curve of degree d as a zero set of $f(x,y,h)$ is now being viewed as a zero set of a complex polynomial $F(z,\overline{z},h)$. The connection between f and F is covered by

Theorem 1. *Let* $f(x,y,h) = \sum\limits_{k+l\leq d} A_{kl}x^k y^l h^{d-k-l}$ *be a curve of degree* d. *The introduction of* $z = x+iy$ *permits a substitution of* x *by* $\frac{z+\overline{z}}{2}$ *and* y *by* $\frac{z-\overline{z}}{2i}$ *in* f. *The result is a polynomial* $F(z,\overline{z},h)$ *depending solely on the variables* z, $\overline{z}$ *and* h. *It may be written in the form*

$$F(z,\overline{z},h) = \sum_{k+l=0}^{d} C_{kl} \cdot z^k \overline{z}^l h^{d-k-l} \tag{8}$$

with complex coefficients C_{kl}. *Using* $p! = 0$, *whenever* $p < 0$, *the connection between the coefficients* A_{kl} *and* C_{kl} *is given by*

$$C_{kl} = \frac{1}{2^{k+l}} \sum_{r=0}^{k+l} \sum_{s=0}^{r} i^{2s-r} \binom{k+l-r}{l-s} \binom{r}{s} A_{(k+l-r)r} \quad . \tag{9}$$

Proof. We have

$$F(z,\overline{z},h) = f\left(\frac{z+\overline{z}}{2}, \frac{z-\overline{z}}{2i}, h\right) = \sum_{k+l=0}^{d} A_{kl} \cdot \left(\frac{z+\overline{z}}{2}\right)^k \left(\frac{z-\overline{z}}{2i}\right)^l h^{d-k-l}$$

The right side of this equation may be expanded using the generalized binomial formula. Changing summation indices and collecting the coefficients of all terms containing a certain product $z^k\overline{z}^l$ we end up with an expression for C_{kl} depending on A. Carrying out these calculations we get exactly Equation (9). □

The complex curve coefficients C_{kl} may be collected and written as parameter vector C. The variable vector now containing products of powers of z, $\overline{z}$ and h will be named Z. Thus $F(z,\overline{z},h) = C^T Z$.

The above theorem states that the linear transformation V in (7) induces a linear transformation, say W^{-T}, acting on the parameter vector A:

$$W^{-T}A =: C \quad .$$

W^{-T} itself is determined by Equation (9). The most important feature of the structure of (9) is that

$$C_{kl} = \overline{C_{lk}} \quad \text{and} \quad C_{kk} = \overline{C_{kk}} \in \mathbb{R} \quad . \tag{10}$$

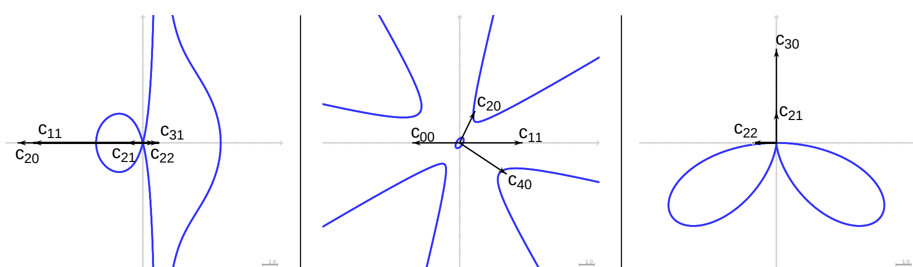
An extraction of this relation form Equation (9) is very technical. We omit this here but instead we introduce the transformation matrix, relating the coefficients for a curve of degree three. This matrix is also directly extracted from (9).

$$\left(\begin{array}{c} C_{00} \\ \hline C_{10} \\ C_{01} \\ \hline C_{20} \\ C_{11} \\ C_{02} \\ \hline C_{30} \\ C_{21} \\ C_{12} \\ C_{03} \end{array}\right) = \left(\begin{array}{c|cc|ccc|cccc} 1 & & & & & & & & & \\ \hline & 1 & -i & & & & & & & \\ & 1 & i & & & & & & & \\ \hline & & & 1 & -i & -1 & & & & \\ & & & 2 & 0 & 2 & & & & \\ & & & 1 & i & -1 & & & & \\ \hline & & & & & & 1 & -i & -1 & i \\ & & & & & & 3 & -i & 1 & -3i \\ & & & & & & 3 & i & 1 & 3i \\ & & & & & & 1 & i & -1 & -i \end{array}\right) \left(\begin{array}{c} A_{00} \cdot 2^{0} \\ \hline A_{10} \cdot 2^{-1} \\ A_{01} \cdot 2^{-1} \\ \hline A_{20} \cdot 2^{-2} \\ A_{11} \cdot 2^{-2} \\ A_{02} \cdot 2^{-2} \\ \hline A_{30} \cdot 2^{-3} \\ A_{21} \cdot 2^{-3} \\ A_{12} \cdot 2^{-3} \\ A_{03} \cdot 2^{-3} \end{array}\right).$$

In the above matrix we can directly see the constraints (10). These are also exactly the conditions a complexified curve must satisfy so that it represents a real algebraic curve. One can easily prove this by counting the degrees of freedom on the curve describing polynomials.

Thus the complexified representation of a curve contains redundant information. Figure 1 shows two curves. The curve parameters are indicated as vectors. However the displayed vectors are suitably scaled and thus we use the label c instead of C.

A similar complexification approach to (7) is used by Jean-Philippe Tarel and David B. Cooper in [5]. They used complex curve representation for pose estimation and invariant recognition in the field of computational shape recognition.



$c^T = \left(0|0\ 0|-8\ -7\ -8| \right.$ $\left. |0\ -1\ -1\ 0|0\ \frac{3}{4}\ 1\ \frac{3}{4}\ 0 \right)$

$c^T = \left(-3|0\ 0|1+2i\ 4\ 1-2i| \right.$ $\left. |0\ 0\ 0\ 0|3-2i\ 0\ 0\ \ 3+2i \right)$

$c^T = \left(0|0\ 0|0\ 0\ 0| \right.$ $\left. |6i\ 2i\ -2i\ -6i|0\ 0\ -1.5\ 0\ 0 \right)$

Fig. 1. Curves determined by complex parameter vectors

2.3 Effects of the complexification

The main benefit of the above reparametrization is that in this representation a rotation of the plane is very easy to describe. A rotation of the plane is a mapping, where

$$z = x + iy \longmapsto \widetilde{z} = \widetilde{x} + i\widetilde{y} = (cx - sy) + i(sx + cy) = (c + is)(x + iy) = e^{i\varphi} z$$

and consequently $\overline{z} \mapsto e^{-i\varphi}\overline{z}$.

This means that $z^k\overline{z}^l \mapsto e^{i(k-l)\varphi}z^k\overline{z}^l$. Thus rotating a curve $F(z, \overline{z}, h) = 0$ by φ we get

$$\widehat{F}_\varphi(z, \overline{z}, h) = \widehat{F}(z, \overline{z}, h) = F\left(e^{-i\varphi}z, e^{i\varphi}\overline{z}, h\right) = \sum_{k+l=0}^{d} C_{kl}e^{i(-k+l)\varphi}z^k\overline{z}^l h^{d-k-l} = 0$$

By the above observations we have already shown

Theorem 2. *Let F be a curve in complex representation with coefficients C_{kl} as shown in Equation (8). A rotation of F around the origin by an angle φ is performed by the mapping*

$$C_{kl} \longmapsto e^{i(-k+l)\varphi}C_{kl} \quad . \tag{11}$$

$\widehat{F}(z, \overline{z}, h) = \sum_{k+l\le d} e^{i(-k+l)\varphi}C_{kl}\, z^k\overline{z}^l h^{d-k-l}$ *is the by φ rotated copy of F.*

Using matrix-vector notation for the transition shown in (11) we have only entries of the form $e^{i\cdot n\varphi}$ with $n \in \mathbb{Z}$:

$$\left(C_{00}|C_{10}\ C_{01}|C_{20}\ C_{11}\ C_{02}|C_{30}\ C_{21}\ C_{12}\ C_{03}|\dots\right)^T \mapsto R_W^{-T}C =$$

$$\underbrace{\left(\begin{array}{c|cc|ccc|cccc|c}
1 & & & & & & & & & & \\ \hline
 & e^{-i\cdot\varphi} & & & & & & & & & \\
 & & e^{i\cdot\varphi} & & & & & & & & \\ \hline
 & & & e^{-i\cdot 2\varphi} & & & & & & & \\
 & & & & 1 & & & & & & \\
 & & & & & e^{i\cdot 2\varphi} & & & & & \\ \hline
 & & & & & & e^{-i\cdot 3\varphi} & & & & \\
 & & & & & & & e^{-i\cdot\varphi} & & & \\
 & & & & & & & & e^{i\cdot\varphi} & & \\
 & & & & & & & & & e^{i\cdot 3\varphi} & \\ \hline
 & & & & & & & & & & \ddots
\end{array}\right)}_{=R_W^{-T}}
\left(\begin{array}{c}
C_{00} \\ \hline C_{10} \\ C_{01} \\ \hline C_{20} \\ C_{11} \\ C_{02} \\ \hline C_{30} \\ C_{21} \\ C_{12} \\ C_{03} \\ \hline \vdots
\end{array}\right) \tag{12}$$

$R_W^{-T} = W^{-T}R^{-T}W^T$ is the complexified version of R^{-T} from (6). The matrix R_W^{-T} acts on our complexified curve parameters C_{kl} and is diagonal. Altogether rotating the plane means rotating all C_{kl} with a *rotation speed* of $-k + l$. Some parameters are even left unaltered: exactly those of type C_{kk}.

This transformation matrix is consistent with our complexification process: Conjugate parameter pairs remain conjugate after the transformation and the real C_{kk} stay real. Opposing (12) to the transformation for the coefficients A_{kl} in the real representation, shown in (6), we can directly see, how powerful complexification is.

Figure 2 shows a curve F and by $\varphi = \frac{\pi}{18}$ rotated copies F_φ of F. The coefficients behave according to (11) and (12).

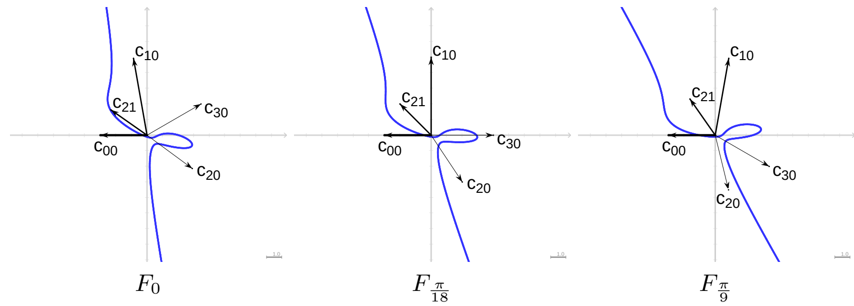


Fig. 2. Rotated curves: c_{00} is stationary, c_{10} and c_{21} rotate by $-\varphi$, c_{20} by -2φ and c_{30} by -3φ.

2.4 Invariants and normal form with respect to rotations around the origin

When a curve is rotated around the origin, curve parameters C_{kk} remain unchanged - invariant so to say. But there are more expressions in the curve parameters, which remain unaltered. For example $C_{10} \cdot C_{01} \mapsto e^{-i\varphi}C_{10} \cdot e^{i\varphi}C_{01} = C_{10} \cdot C_{01}$.

Any product of curve parameters is multiplied by $e^{ir\varphi}$ (for suitable r), when the plane is rotated by φ around the origin. This is likewise a rotation. The rotation speed r of such a product is the sum of the rotation speeds of factors of the product. For example

$$C_{kl} \cdot C_{pq} \longmapsto \exp(i \underbrace{[(-k+l)+(-p+q)]}_{=r} \varphi) \cdot C_{kl} \cdot C_{pq} = e^{ir\varphi} \cdot C_{kl} \cdot C_{pq} \quad .$$

The rotation speed of C_{kl} is $-k+l$, the one of C_{pq} is $q-p$ and the rotation speed of the product $C_{kl} \cdot C_{pq}$ is $r = (-k+l)+(-p+q)$.

We have listed a few C_{kl} with corresponding rotation speeds in the following table. Coefficients C_{kl} with the same index-difference $k-l$ and thus the same rotation speed $l-k$ are in the same column. Coefficients of the same block, i. e. coefficients with the same index-sum, are in the same line.

$block_t(C)$	curve coefficients							
$t=0$					C_{00}			
$t=1$				C_{10}		C_{01}		
$t=2$			C_{20}		C_{11}		C_{02}	
$t=3$		C_{30}		C_{21}		C_{12}		C_{03}
...	...	...	...	...	...	...	...	... ...
rotation speed	...	-3	-2	-1	0	1	2	3 ...

(13)

According to our observations any product of coefficients remains unchanged, when the sum of the rotation speeds r evaluates to zero. This gives us a wide range of invariant expressions. We can directly see three types of invariant expressions:

1. form parameters C_{kk}
2. lengths $C_{kl}C_{lk} = |C_{kl}|^2$
3. angle relations contained in $(C_{kl})^{p-q}(C_{pq})^{-k+l}$

Assume that for an arbitrary curve $\mathcal{C}_f$ all values to all well defined expressions in 1, 2 and 3 are given. Using these, a curve g can be constructed exhibiting these same values: We can use the coefficients C_{kk} directly for g. The $C_{kl}C_{lk}$ give us the lengths of the remaining coefficients C_{kl} and the $(C_{kl})^{p-q}(C_{pq})^{-k+l}$ give us relations between the coefficients. Using these coefficients of g are determined up to one degree of freedom. Then this reconstructed $\mathcal{C}_g$ will necessarily be a rotated copy of f. Thus f is reconstructible up to rotations around the origin. We do not go into further details here. Instead we focus on normal forms.

If we have a curve whose only non-vanishing coefficients are of the form C_{kk}, the curve is not affected by rotations around the origin. Thus the whole curve is rotational invariant. Therefore it must consist of one or several circles centered at the origin. These curves are of the form $\sum_{k=0}^{\frac{d}{2}} C_{kk} z^k \overline{z}^k h^{d-2k} = 0$, which of course is already a normal form with respect to rotations around the origin.

If there is a $C_{kl} \neq 0$ with $k \neq l$, then the curve is affected by rotations. Let S be the set of all $C_{kl} \neq 0$ with $k \neq l$ for a given curve f. Taking a coefficient $C_{kl} \in S$ we can find angles by which the curve has to be rotated such that the transformed curve $\widehat{f}$ has a positive real $\widehat{C_{kl}} \in \mathbb{R}_{>0}$. In general there is a whole set of angles with this property: The coefficient C_{kl} transforms under rotations by φ according to $\widetilde{C_{kl}} = C_{kl}e^{i\varphi(-k+l)}$. Thus φ is determined up to the $(k-l)$-th roots of unity and

$$e^{i\varphi} = \left(\frac{C_{kl}}{|C_{kl}|}\right)^{\frac{1}{k-l}} \cdot u \quad \text{with} \quad u \in \left\{z \in \mathbb{C} \;\middle|\; z^{k-l} = 1\right\} \quad .$$

Now, rotations partition the set of curves into classes of rotational equivalent curves. A representation system and thus a normal form with respect to rotations is given by

Theorem 3. *Let g be an arbitrary curve with coefficients $C_{kl} = r_{kl}e^{i\varphi_{kl}}$. Let $S = S_g$ be the set of all $C_{kl} \neq 0$ with $k \neq l$. Let the order on S be $\succ$ as defined in Definition 2. Without loss of generality we assume that all angles φ_{kl} are contained in $[0, 2\pi)$ at any time. Let $\Phi = \Phi_0 = [0, 2\pi)$. The following algorithm defines a unique curve in each equivalence class of rotational equivalent curves: Take the largest coefficient $C_{pq} \in S$ with respect to $\succ$. Determine $\Phi_1 \subset \Phi_0$ such that each $\varphi \in \Phi_1$ would give us a $\widehat{C_{pq}} = \widehat{r_{pq}}e^{i\widehat{\varphi_{pq}}} = C_{pq}e^{i\varphi(-p+q)}$ with minimal $\widehat{\varphi_{pq}} \in [0, 2\pi)$. If $|\Phi_1| = 1$, then rotate the curve by the sole $\varphi \in \Phi_1$. Else take the next largest coefficient in S and determine a corresponding $\Phi_2 \subseteq \Phi_1$. Proceed as before till a Φ_n has only one element or all coefficients in S have been taken. If there is no next largest coefficient in S because any $C_{kl} \in S$ has been considered, then each rotation by any φ in the corresponding set results in the same curve.*

Proof. The theorem introduces a minimization problem on the angles φ_{kl} of the ordered coefficients $C_{kl} \in S$. Each curve g of degree d may be associated with a coefficient vector and therefore with an *angle vector*. For $n = \frac{(d+1)(d+2)}{2}$ this is $\alpha_g = (\varphi_{00}, \varphi_{10}, ..., \varphi_{0d}) \in [0, 2\pi)^n$. On the set of all such possible vectors of the same length a total order may be introduced: $(\varphi_{00}, \varphi_{10}, ..., \varphi_{0d}) < (\psi_{00}, \psi_{10}, ..., \psi_{0d})$, if $\varphi_{0d} < \psi_{0d}$ or $\varphi_{0d} = \psi_{0d} \ \wedge \ \varphi_{1,d-1} < \psi_{1,d-1}$, or ... . The order by which the angles φ_{kl} are considered is given by $\succ$.

Taking the rotations of the plane, they introduce not only a partition of the set of curves of degree d but also a partition of the angle vectors. Thus a representation system of the angle vectors with respect to rotations of the plane is directly associated with a representation system of curves with respect to rotations. Now, given a curve, we may determine the rotation-angles φ such that the angle φ_{kl} of the first coefficient C_{kl} in $(S, \succ)$ becomes minimal, then the angle of the second largest and so on. This corresponds to the minimization of the angle vector α_g of a curve g with respect to $<$ using only rotations. Thereby the set of possible rotation angles Φ_r becomes successively smaller. In the end we get a unique curve with minimal angle vector but not necessarily a set Φ_n of cardinality one. If we end up with multiple angles, then all rotations lead to the same minimal angle vector and thus the same curve.

Altogether, rotations minimizing the angle vector rotate a given curve into canonical form and the above theorem introduces a rotational normal form for curves. □

If we have different rotation angles $\varphi \in [0, 2\pi)$ such that rotations by these angles result in the same curve, then the given curve must have been rotational symmetric. More on rotational symmetry is covered in sections 4 and 5. As an example for a curve in rotational normal form see Figure 2. There $\max((S, \succ)) = C_{30}$. Only a rotation by $\varphi = \frac{\pi}{18}$ gives us a positive real $\widehat{C_{30}}$. Thus the curve $\widehat{F}(z, \overline{z}, h) = F_{\frac{\pi}{18}}(z, \overline{z}, h) = 0$ is in normal form.

3 Translation

3.1 Effects of translations on complexified curve parameters

When talking about translations it is convenient to use dehomogenized coordinates or to scale the homogenizing component to $h = 1$. Thus a point p given by $p = (z, \overline{z}, h)$ with $h = 1$ is translated by t, when

$$\begin{pmatrix} z \\ \overline{z} \\ 1 \end{pmatrix} \longmapsto \begin{pmatrix} 1 & 0 & t \\ 0 & 1 & \overline{t} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} z \\ \overline{z} \\ 1 \end{pmatrix} \quad . \tag{14}$$

If $h \neq 1$, then t would be a scaled Euclidean translation parameter. We represent curves by explicitly using multinomial coefficients $m_{kl} = m_{kl,d} = \frac{d!}{k! \cdot l! \cdot (d-k-l)!}$ as scalar factors to the coefficients. For a degree d curve $\mathcal{C}_f$ we choose new parameters c_{kl} such that

$$m_{kl} \cdot c_{kl} = C_{kl} \quad . \tag{15}$$

Clearly, the new coefficients c_{ij} can be considered as parameters of $\mathcal{C}_f$ as well as the C_{ij}.

Theorem 4. *Let F be a curve defined by $F(z,\overline{z},h)=0$. The by t translated copy of F is $\widetilde{F}_t=\widetilde{F}$. Both curves shall be represented using multinomial coefficients: $F(z,\overline{z},h)=\sum\limits_{k+l=0}^{d} m_{kl}c_{kl}z^k\overline{z}^l$ and $\widetilde{F}(z,\overline{z},h)=\sum\limits_{k+l=0}^{d} m_{kl}\widetilde{c_{kl}}z^k\overline{z}^l$. Thus*

$$\widetilde{c_{kl}}=\sum_{p=k}^{d-l}\sum_{q=l}^{d-p}(-1)^{(r+s)-(k+l)}\frac{(d-k-l)!}{(p-k)!\,(q-l)!\,(d-p-q)!}t^{p-k}\overline{t}^{q-l}c_{pq} \quad . \tag{16}$$

Proof. Translating F by t, we get

$$0=\widetilde{F}_t(z,\overline{z})=\widetilde{F}(z,\overline{z}):=F(z-t,\overline{z}-\overline{t})=\sum_{k+l=0}^{d} m_{kl}c_{kl}\,(z-t)^k\left(\overline{z}-\overline{t}\right)^l \quad .$$

We may now expand the right side of the above equation by using the generalized binomial formula. Collecting the coefficients of a certain $z^k\overline{z}^l$ and changing the summation-order we get exactly Equation (16). □

Equation (16) completely describes the transformation of the parameters. However, the matrix representation of this transformation is by far more instructive. To visualize the transformation using matrix notation, we choose $d=3$. To avoid the minus signs in the matrix-vector notation, we choose to translate by $-t$ instead of t. Then

$$\left(c_{00}\middle|c_{10}\ c_{01}\middle|c_{20}\ c_{11}\ c_{02}\middle|c_{30}\ c_{21}\ c_{12}\ c_{03}\right)^T \longmapsto$$

$$\left(\begin{array}{c|cc|ccc|cccc}
1 & 3t & 3\overline{t} & 3t^2 & 6t\overline{t} & 3\overline{t}^2 & t^3 & 3t^2\overline{t} & 3t\overline{t}^2 & \overline{t}^3 \\
\hline
 & 1 & & 2t & 2\overline{t} & & t^2 & 2t\overline{t} & \overline{t}^2 & \\
 & & 1 & & 2t & 2\overline{t} & & t^2 & 2t\overline{t} & \overline{t}^2 \\
\hline
 & & & 1 & & & t & \overline{t} & & \\
 & & & & 1 & & & t & \overline{t} & \\
 & & & & & 1 & & & t & \overline{t} \\
\hline
 & & & & & & 1 & & & \\
 & & & & & & & 1 & & \\
 & & & & & & & & 1 & \\
 & & & & & & & & & 1
\end{array}\right)
\left(\begin{array}{c}
c_{00} \\ \hline c_{10} \\ c_{01} \\ \hline c_{20} \\ c_{11} \\ c_{02} \\ \hline c_{30} \\ c_{21} \\ c_{12} \\ c_{03}
\end{array}\right) \quad .$$

We can see a lot of structure in the transformation law (16). The most prominent feature is that the transformation has upper triangular and block structure. Coefficients $\widetilde{c_{kl}}$ depend only on c_{kl} and c_{pq} with $p+q>k+l$. Thereby c_{pq} is multiplied by an expression in t and $\overline{t}$. The degree of this expression is determined by the difference of the summed indices: $(p+q)-(k+l)$. Thus the blocks in the diagonal are identity matrices, first sup-diagonal blocks are linear in t and $\overline{t}$, the second are quadratic and so on. Consequently sub-leading coefficients, i.e. coefficients c_{kl} with $k+l=d-1$, change only linearly in t and $\overline{t}$ when the plane is translated. Clearly, leading coefficients do not change at all.

3.2 Annihilating coefficients by curve-translation

Now we can exploit the linear dependence of sub-leading coefficients on t and $\bar{t}$. We use it to find a translation such that a transformed sub-leading coefficient vanishes. The resulting curve is independent of any previous translations - at least if it satisfies the precondition of the theorem below. With this ingredient at hand we can build up translatorial normal forms later.

The process of translating a curve such that some $\widetilde{c}_{kl} = 0$ with $k + l = d - 1$, will be called annihilation. By (16) we have

$$\widetilde{c_{kl}} = c_{kl} - tc_{(k+1)l} - \bar{t}c_{k(l+1)} \quad . \tag{17}$$

For convenience and clarity we omit the indices by substituting

$$\mathfrak{c} = r_{\mathfrak{c}} e^{i\gamma} = c_{kl}, \qquad \mathfrak{a} = r_{\mathfrak{a}} e^{i\alpha} = c_{(k+1)l} \qquad \text{and} \qquad \mathfrak{b} = r_{\mathfrak{b}} e^{i\beta} = c_{k(l+1)} \quad .$$

In what follows $\mathfrak{c}$ will always be a subleading coefficient. We get

Theorem 5. *Let $r_{\mathfrak{a}} \neq r_{\mathfrak{b}}$ and $\Delta = r_{\mathfrak{a}}^2 - r_{\mathfrak{b}}^2$. The by*

$$t = \frac{1}{\Delta}\left(\mathfrak{c}\,\bar{\mathfrak{a}} - \bar{\mathfrak{c}}\,\mathfrak{b}\right) \tag{18}$$

translated copy $\widetilde{F} = \widetilde{F}_t$ of F has a vanishing subleading coefficient $\widetilde{\mathfrak{c}} = 0$. Translations by a parameter different from t in Equation (18) imply $\widetilde{\mathfrak{c}} \neq 0$.

Proof. We have $\widetilde{\mathfrak{c}} = 0 = \mathfrak{c} + t\mathfrak{a} + \bar{t}\mathfrak{b}$ from the transformation law for subleading coefficients (17). We split the participating coefficients into real and imaginary parts:

$$\mathfrak{a} = x_{\mathfrak{a}} + iy_{\mathfrak{a}}, \quad \mathfrak{b} = x_{\mathfrak{b}} + iy_{\mathfrak{b}} \quad \text{and} \quad \mathfrak{c} = x_{\mathfrak{c}} + iy_{\mathfrak{c}} \quad .$$

Together with $t = x_t + iy_t$ we get

$$x_{\mathfrak{c}} iy_{\mathfrak{c}} = [(x_{\mathfrak{a}} + x_{\mathfrak{b}})x_t + (-y_{\mathfrak{a}} + y_{\mathfrak{b}})y_t] + i\,[(y_{\mathfrak{a}} + y_{\mathfrak{b}})x_t + (x_{\mathfrak{a}} - x_{\mathfrak{b}})y_t]$$

or in matrix-vector notation

$$\begin{pmatrix} x_{\mathfrak{c}} \\ y_{\mathfrak{c}} \end{pmatrix} = \begin{pmatrix} x_{\mathfrak{a}} + x_{\mathfrak{b}} & -y_{\mathfrak{a}} + y_{\mathfrak{b}} \\ y_{\mathfrak{a}} + y_{\mathfrak{b}} & x_{\mathfrak{a}} - x_{\mathfrak{b}} \end{pmatrix} \begin{pmatrix} x_t \\ y_t \end{pmatrix} = M \begin{pmatrix} x_t \\ y_t \end{pmatrix} \quad . \tag{19}$$

In order to be able to uniquely solve for t, $\det(M)$ needs to be non-zero. But

$$\det(M) = x_{\mathfrak{a}}^2 - x_{\mathfrak{b}}^2 + y_{\mathfrak{a}}^2 - y_{\mathfrak{b}}^2 = r_{\mathfrak{a}}^2 - r_{\mathfrak{b}}^2 = \Delta \neq 0 \quad .$$

Thus by the preconditions of our theorem: M is invertible and

$$\begin{pmatrix} x_t \\ y_t \end{pmatrix} = \frac{1}{\Delta} \begin{pmatrix} x_{\mathfrak{a}} - x_{\mathfrak{b}} & y_{\mathfrak{a}} - y_{\mathfrak{b}} \\ -y_{\mathfrak{a}} - y_{\mathfrak{b}} & x_{\mathfrak{a}} + x_{\mathfrak{b}} \end{pmatrix} \begin{pmatrix} x_{\mathfrak{c}} \\ y_{\mathfrak{c}} \end{pmatrix} \quad .$$

Consequently,

$$\begin{aligned} t &= \frac{1}{\Delta}\left[(x_{\mathfrak{a}} - x_{\mathfrak{b}})x_c + (y_{\mathfrak{a}} - y_{\mathfrak{b}})y_c + i(-y_{\mathfrak{a}} - y_{\mathfrak{b}})x_c + i(x_{\mathfrak{a}} + x_{\mathfrak{b}})y_c\right] \\ &= \frac{1}{\Delta}\left(\mathfrak{c}\bar{\mathfrak{a}} - \bar{\mathfrak{c}}\mathfrak{b}\right) \quad . \end{aligned}$$

Thus a translation by t gives the desired result. □

In the above theorem the translation making a curve coefficient vanish was unique. This was ensured by the precondition $r_{\mathfrak{a}} \neq r_{\mathfrak{b}}$. In case of $r_{\mathfrak{a}} = r_{\mathfrak{b}}$, there still might exist a translation of the given curve, annihilating $\mathfrak{c}$.

Theorem 6. *Let $r_{\mathfrak{a}} = r_{\mathfrak{b}} \neq 0$. Let $\widetilde{g} = \widetilde{g}_t$ be the by t translated copy of g.*

There is a $t \in \mathbb{C}$ with $\widetilde{\mathfrak{c}} = 0$ if and only if already $\mathfrak{c} = 0$ or $\gamma \in \tau + \pi\mathbb{Z}$ with $\tau = \frac{\alpha+\beta}{2}$. Either way t is not unique.

Proof. Annihilating $\mathfrak{c}$ means finding a translation parameter $t = r_t e^{i\vartheta}$ with $\mathfrak{c} = t\mathfrak{a} + \bar{t}\mathfrak{b}$. Let us look at the set L of all points $z \in \mathbb{C}$ with $z = t\mathfrak{a} + \bar{t}\mathfrak{b}$ for an arbitrary $t \in \mathbb{C}$. We have

$$\begin{aligned} z &= r_{\mathfrak{a}} r_t \left(e^{i(\alpha+\vartheta)} + e^{i(\beta-\vartheta)} \right) = r_{\mathfrak{a}} r_t \left(e^{i(\frac{\alpha-\beta}{2}+\vartheta)} + e^{i(\frac{-\alpha+\beta}{2}-\vartheta)} \right) e^{i\frac{\alpha+\beta}{2}} \\ &= r_{\mathfrak{a}} r_t \cdot 2\cos\left(\frac{\alpha-\beta}{2} + \vartheta \right) e^{i\frac{\alpha+\beta}{2}} \quad . \end{aligned}$$

Thus L is a line through the origin with an angle of $\tau = \frac{\alpha+\beta}{2}$ to the real axis. If and only if $\mathfrak{c}$ is contained in L, then a parameter t exists such that $\widetilde{\mathfrak{c}} = 0$. This means that $\mathfrak{c} = 0$ or $\gamma \in \tau + \pi\mathbb{Z}$ must hold.

Let $\mathfrak{c} \neq 0$. W.l.o.g. $\mathfrak{c} = r_{\mathfrak{c}} e^{i\tau}$ with $r_{\mathfrak{c}} \in \mathbb{R}_{\geq 0}$. Otherwise substitute $r_{\mathfrak{c}}$ by $-r_{\mathfrak{c}}$ in the formula below. Then equation $\mathfrak{c} = t\mathfrak{a} + \bar{t}\mathfrak{b}$ transforms to

$$r_{\mathfrak{c}} e^{i\tau} = r_{\mathfrak{a}} r_t \cdot 2\cos\left(\frac{\alpha-\beta}{2} + \vartheta \right) e^{i\tau}$$

and consequently for $\vartheta \notin \frac{-\alpha+\beta+\pi}{2} + \pi\mathbb{Z}$

$$r_t = \frac{r_{\mathfrak{c}}}{2 r_{\mathfrak{a}} \cos\left(\frac{\alpha-\beta}{2} + \vartheta \right)} \quad .$$

Choosing any such ϑ gives us a corresponding $t = r_t e^{i\vartheta}$ making $\widetilde{\mathfrak{c}} = 0$ vanish.

If $\mathfrak{c} = 0$, then $t = \lambda e^{i\frac{-\alpha+\beta+\pi}{2}}$ for any $\lambda \in \mathbb{R}$ gives us $\widetilde{\mathfrak{c}} = 0$.

Thus a translation, which annihilates $\mathfrak{c}$, is not unique. □

Let $\mathfrak{c} = c_{kl}$ be a subleading coefficient and $\mathfrak{a} = c_{k+1,l}$ as well as $\mathfrak{b} = c_{k,l+1}$ such that $r_{\mathfrak{a}} \neq r_{\mathfrak{b}}$. Then the corresponding curve g can be translated such that the translated copy $\widetilde{g}$ of g has a vanishing $\widetilde{\mathfrak{c}} = 0$. The corresponding translation-parameter is $t = \frac{\mathfrak{c}\bar{\mathfrak{a}} - \bar{\mathfrak{c}}\mathfrak{b}}{r_{\mathfrak{a}}^2 - r_{\mathfrak{b}}^2}$. Before we continue with translatorial normal forms, we study

3.3 Annihilation in the case of conics

We exemplify the process of annihilation with conics. The knowledge from the theory of conics may help to understand. There are many interesting theoretical parallels here. We focus on the complexified representation and annihilation. The treatment of curves of higher order is in principle not very different from the following.

A quadratic equation is given by $q(z,\overline{z}) = c_{00} + 2c_{10}z + 2c_{01}\overline{z} + c_{20}z^2 + 2c_{11}z\overline{z} + c_{02}\overline{z}^2 = 0$. The only subleading coefficients are c_{10} and c_{01}. Because of $c_{10} = \overline{c_{01}}$, annihilating c_{10} means automatically annihilating c_{01} and vice versa. Thus without loss of generality we focus on c_{10} alone.

According to Theorem 5 we can perform annihilation whenever $|c_{20}| \neq |c_{11}|$. Figure 3 shows a conic satisfying this condition. The annihilating translation vector

$$t = \frac{c_{10}c_{20} - c_{01}c_{11}}{c_{20}c_{02} - c_{11}^2}$$

is also indicated. Now, let $|c_{20}| = |c_{11}|$. We assume that $c_{11} > 0$. Otherwise examine $-q$. Then with $c_{20} = r \cdot e^{i\psi}$:

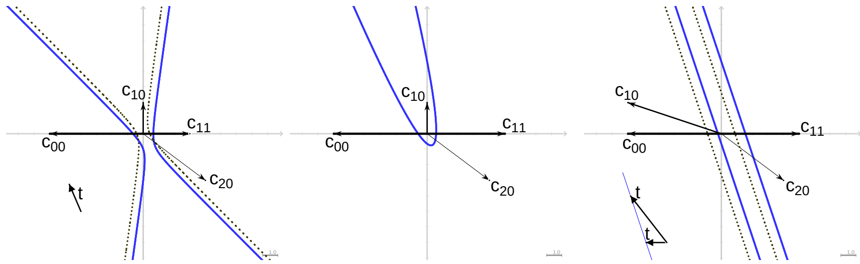
$$q(z,\overline{z}) = c_{00} + 2c_{10}z + 2c_{01}\overline{z} + r(e^{i\frac{\psi}{2}}z + e^{-i\frac{\psi}{2}}\overline{z})^2 = 0 \quad .$$

Let us look at the projective extension of the complex plane - to be more precise: at the restriction of q to the line at infinity, namely $r(e^{i\frac{\psi}{2}}z + e^{-i\frac{\psi}{2}}\overline{z})^2 = 0$. There we can see a double root at $z = e^{-i\frac{\psi}{2}}$. From the theory of conics we know that in this case we must have parabolas or pairs of parallel lines. In Figure 3 both cases are shown. Looking back to the case where $|c_{20}| \neq |c_{11}|$, we must have had hyperbolas, ellipses, circles or intersecting line pairs. There, the linear (subleading) part of the equation could be annihilated by translations which is confirmed by the theory of quadratic curves.

If $c_{10} = \lambda e^{i\frac{\psi}{2}}$ for some $\lambda \in \mathbb{R}$, then annihilation is possible by Theorem 6 despite $|c_{20}| = |c_{11}|$. In this case

$$q(z,\overline{z}) = \left(\alpha + \sqrt{r}(e^{i\frac{\psi}{2}}z + e^{-i\frac{\psi}{2}}\overline{z})\right) \cdot \left(\beta + \sqrt{r}(e^{i\frac{\psi}{2}}z + e^{-i\frac{\psi}{2}}\overline{z})\right)$$

with $\alpha + \beta = \frac{2\lambda}{\sqrt{r}}$ and $\alpha \cdot \beta = c_{00}$. This means that q degenerates into two parallel lines: q is a product of two terms which are linear in z and $\overline{z}$ and with equal leading coefficients. This again corresponds nicely to the theory of conics. By translation we can achieve that one point of equal distance to the lines is the origin. Consequently $\alpha = -\beta$ and thus $\widetilde{c_{10}} = 0$. As parallel lines have no unique midpoint, such a translation can not be unique. This corresponds to Theorem 6. Figure 3 shows annihilation for parallel lines.



$|c_{20}| \neq |c_{11}|$
normal-form is dotted

$|c_{20}| = |c_{11}|$ and $c_{10} \notin L$

$|c_{20}| = |c_{11}|$ and $c_{10} \in L$
normal-form is dotted
more than one t annihilates

Fig. 3. Conics and annihilation

3.4 Normal form and invariants with respect to translations

We will confine ourselves with curves, where the absolute values $|c_{kl}|$ of all leading coefficients c_{kl} are not equal. The specialty of these curves is that there is a unique translation annihilating a certain subleading coefficient (see Theorem 5). This proofs the following theorem.

Theorem 7. *Let F be an arbitrary curve with leading coefficients of different absolute values. Let $c_{p+1,q}$ and $c_{p,q+1}$ be the largest coefficients with respect to $\succ$ such that $|c_{p+1,q}| \neq |c_{p,q+1}|$. We define: A curve F is in canonical form with respect to translations, if $c_{pq} = 0$. This defines a translatorial normal form in the sense of Definition 1.*

Assume that a curve F is in translatorial normal form according to Theorem 7. Rotating this curve preserves the absolute value of each coefficient, because it changes it only by a factor $e^{ik\varphi}$ for some angle $k\varphi$. Thus transforming F into rotational normal form preserves the translatorial normal form. Taking the two normal-forms together we have a normal form with respect to arbitrary rotations and translations: a normal form with respect to orientation preserving Euclidean transformations.

For curves with leading coefficients of the same absolute value the translation process, transforming a curve into translatorial normal-form, is uniquely determined. Thus the coefficients of the translated curve are invariants of the curve describing equation with respect to translations. The leading coefficients are not affected by translations and thus also invariant. The other coefficients $\widetilde{c_{kl}}$ with $k + l < d$ can be calculated using (16) and have a more complicated structure.

Take a quadric $q(z, \overline{z}) = 0$ for example. Annihilating a subleading coefficient by translation means $\widetilde{c_{10}} = 0$. The translated conic will be

$$\widetilde{q}(z, \overline{z}) = \widetilde{c_{00}} + \widetilde{c_{20}} z^2 + \widetilde{c_{11}} z\overline{z} + \widetilde{c_{02}} \overline{z}^2 \quad .$$

The translation parameter t is $t = \frac{c_{10}\overline{c_{20}} - \overline{c_{10}}c_{11}}{|c_{20}|^2 - |c_{11}|^2}$ according to Theorem 5. Thus by the transformation law (16) we get (with $\Delta = c_{20}c_{02} - c_{11}^2$):

$$\begin{aligned}
\widetilde{c_{00}} = q(t,\bar{t}) &= c_{00} + \frac{1}{\Delta}\left(2c_{10}c_{01}c_{11} - c_{10}^2 c_{02} - c_{01}^2 c_{20}\right) \\
&= \frac{1}{\Delta}\det\begin{pmatrix} c_{20} & c_{11} & c_{10} \\ c_{11} & c_{02} & c_{01} \\ c_{10} & c_{01} & c_{00} \end{pmatrix} = \frac{\det(M)}{\Delta} \quad ,
\end{aligned}$$

$$\widetilde{c_{10}} = 0, \quad \widetilde{c_{01}} = 0,$$

$$\widetilde{c_{20}} = c_{20}, \quad \widetilde{c_{11}} = c_{11} \quad \text{and} \quad \widetilde{c_{02}} = c_{02} \quad .$$

We have the following invariants of the quadratic equation: $\widetilde{c_{00}}$, $\widetilde{c_{20}} = \overline{\widetilde{c_{02}}}$ and $\widetilde{c_{11}}$.

Therefore $\Delta = |c_{20}|^2 - {c_{11}}^2$ is invariant. Thus $\Delta\widetilde{c_{00}} = \det(M)$ is also invariant. $\det(M)$ is even a projective invariant of the quadratic equation.

Invariants of the conic as curve can be obtained by normalizing the coefficients. This can be done by dividing the coefficients by the Frobenius norm $\|c\|_F$ of the coefficient vector c. Thus $\frac{\det(M)}{\|c\|_F^3}$ and $\frac{\Delta}{\|c\|^2}$ are invariants of the curve. If $\det(M) \neq 0$ the same is true for the combination $\frac{\Delta^3}{\det(M)^2}$.

It is instructive to analyze what this specific invariant means. For this consider the very simple quadric given by $ax^2 + by^2 - 1 = 0$. If both a and b are positive, then this conic is an ellipse, whose axes of symmetry are aligned to the coordinate system. In this case the complex representation turns out to be $\frac{a-b}{4}z^2 + \frac{a+b}{2}z\overline{z} + \frac{a-b}{4}\overline{z}^2 - 1$. Thus we have $c_{20} = c_{02} = \frac{a-b}{4}$, $c_{11} = \frac{a+b}{4}$ (remember we have multinomial pre-scaling), $c_{10} = 0$ and $c_{00} = -1$. We want to study what happens if we insert these coefficients into $\frac{\Delta^3}{\det(M)^2}$. First observe that the numerator as well as the denominator are of homogeneous degree 6 in the c_{ij}. Thus we may multiply all coefficients by 4 without altering the value of the expression. So we get:

$$\frac{\det\begin{pmatrix} a-b & a+b \\ a+b & a-b \end{pmatrix}^3}{\det\begin{pmatrix} a-b & a+b & 0 \\ a+b & a-b & 0 \\ 0 & 0 & -4 \end{pmatrix}^2} = -\frac{ab}{4}.$$

This amazingly simple expression is nothing else but $-(\Omega \cdot \frac{1}{2\pi})^2$, where $\Omega = \pi\sqrt{ab}$ is the area of our ellipse. Thus we have a Euclidean explanation for our expression being invariant: It is just a function of the area!

4 Rotation symmetry with origin as center

Understanding what happens to a curve parameter vector under rotations around the origin and translations, we can now study rotation symmetry. When we speak of rotation symmetry we always mean symmetric with respect to rotations around the origin unless stated otherwise. Curves consisting only of circles concentric to the origin are invariant under rotations. We can identify these curves by checking whether the c_{kk} coefficients are the only non-vanishing parameters.

Let a dehomogenized curve be given by $F(z, \overline{z}) = 0$. Rotating the curve around the origin by an angle ψ creates another curve $F_\psi(z, \overline{z}) = \widehat{F}(z, \overline{z}) = 0$. F is said to be rotational symmetric if there is an angle $\psi \notin 2\pi\mathbb{Z}$ such that the zero sets of F and the by ψ rotated curve $\widehat{F}$ coincide. In case of $F_\psi = F$ we will call ψ a symmetry angle. This way $\psi = 2\pi$ is also a symmetry angle. But if $\psi = k \cdot 2\pi$ with $k \in \mathbb{Z}$ are the only symmetry angles for a curve F, then F is not rotational symmetric. Clearly, if ψ is a symmetry angle, then $u \cdot \psi$ ($u \in \mathbb{Z}$) is also a symmetry angle. If ψ_1 and ψ_2 are two symmetry angles, then any linear combination of both also has this property.

Without loss of generality we may assume that the polynomial $F(z, \overline{z})$ is an irreducible polynomial. When curves with multiple components are of interest, each single component may be examined. (For a treatment on irreducible polynomials and components of curves see [1].) We have:

Lemma 1. *Let F be a rotational symmetric algebraic curve. Then any symmetry angle ψ is a rational multiple of 2π or F is a circle.*

Proof. We interpret the origin as a circle with zero radius. If the curve is a circle, then any ψ is a symmetry angle. Assume that ψ is an irrational multiple of 2π. Let F be arbitrary of degree d but not a circle. Then there is a point z on the curve which is not the origin. The images of z under iterated rotations by ψ build a set of infinitely many different points on a circle c_z. By assumption $F \neq c_z$ and consequently F must intersect c_z in infinitely many points. But F was algebraic and therefore of finite degree d. Thus the number of intersections of F and c_z is at most $2 \cdot d$ by Bezout's theorem (cf. [1]). □

Lemma 2. *If F is rotational symmetric and not a circle, then the smallest positive symmetry angle ψ may be written in the form $\psi = \frac{2\pi}{u}$ with $u \in \mathbb{N}$.*

Proof. By Lemma 1: $\psi = \frac{p}{q} \cdot 2\pi$ and without loss of generality $p, q \in \mathbb{N}$ and $\gcd(p, q) = 1$. If $p = 1$ nothing is to be shown. Otherwise there are $m, n \in \mathbb{Z}$ with $mp + nq = 1 = \gcd(p, q)$. If ψ is a symmetry angle then for any $l \in \mathbb{Z}$ the angles $m\psi + l \cdot 2\pi$ have this property, too. Let $l = n$. Then

$$m \cdot \frac{p}{q} \cdot 2\pi + n \cdot \frac{q}{q} \cdot 2\pi = \frac{mp + nq}{q} \cdot 2\pi = \frac{1}{q} \cdot 2\pi < \psi \quad .$$

Having more than one symmetry angle $\psi_i = \frac{p_i}{q_i} \cdot 2\pi$, we know that all angles $\frac{1}{q_i} \cdot 2\pi$ are symmetry angles. u equal to the least common multiple of the q_i proves the lemma. □

Lemma 2 says that F has an u-fold rotation symmetry with respect to rotations around the origin.

Now, let F be rotational symmetric and $\widetilde{F}$ be a rotated copy of F with coinciding zero sets. We assume that F is irreducible. Otherwise we would have to decompose $F(z, \overline{z})$ and apply the following theory to all components. In case of irreducibility, by Study's lemma (cf. [1]) there is a $\lambda \in \mathbb{C}$ with

$$F(z, \overline{z}) = \lambda \cdot \widetilde{F}(z, \overline{z}), \quad \forall (z, \overline{z}) \quad .$$

By Lemma 2

$$F(z, \overline{z}) = \lambda \cdot F(e^{i\frac{2\pi}{u}}\ z, e^{-i\frac{2\pi}{u}}\ \overline{z}), \quad \forall (z, \overline{z}) \quad . \tag{20}$$

The aim is now to determine the largest $u \in \mathbb{Z}$ satisfying the above Equation (20). This equation holds only if there is a $\lambda \in \mathbb{C}$ such that for all coefficients c_{pq} of F

$$c_{pq} = \lambda \cdot c_{pq} \cdot e^{\frac{2\pi}{u}(p-q)} \quad . \tag{21}$$

Vanishing coefficients obviously impose no restriction because (21) is satisfied for all λ and all u. If $c_{pq} \neq 0$, then $\lambda = e^{\frac{2\pi i}{u}(q-p)}$. In case of $p = q$ we have $\lambda = 1$.

Let us examine the restrictions imposed on u by two non-vanishing parameters $c_{kl} \neq 0$ and $c_{pq} \neq 0$. Because of λ in equation (21) we get

$$\begin{aligned} &\exp\left(i\tfrac{2\pi}{u}(l-k)\right) = \exp\left(i\tfrac{2\pi}{u}(q-p)\right) \\ \Leftrightarrow\ &\tfrac{2\pi}{u}(l-k) \equiv \tfrac{2\pi}{u}(q-p) \mod 2\pi \\ \Leftrightarrow\ &\exists v \in \mathbb{Z} : v \cdot u = (q-p) - (l-k)\ = r_{klpq} \end{aligned}$$

Thus u must be a divisor of $r_{klpq} = (q-p) - (l-k)$. This means that the differences r_{klpq} of the rotation speeds $(q-p)$ and $(l-k)$ of any pair of non-vanishing coefficients c_{kl} and c_{pq} determine the rotation symmetry: Let R be the set of all such r_{klpq}, then u must divide any element of R. In case a $c_{kl} \neq 0$ we know that $c_{lk} = \overline{c_{kl}} \neq 0$. Thus not only differences but also sums of rotation speeds matter. Consequently R happens to be equal to $R = \{0\}$ exactly when the only non vanishing parameters are of the form c_{kk}. Thus $R = \{0\}$ only for unions of circles and according to the above calculations any u is possible. Otherwise if $R \neq \{0\}$, the maximal u is $\gcd(R)$ and g is $\gcd(R)$-fold rotational symmetric. This proves

Theorem 8. *Let g be a curves and let R be the set of all differences in the rotation speeds of all pairs of non-vanishing coefficients. (For rotation speeds of coefficients compare with (13).) g is u-fold rotational symmetric if*

$$u \mid r \quad \forall r \in R \quad .$$

To determine $\gcd(R)$ and thus a maximal u, we can look at the coefficient triangle in (13). We color a column whenever it contains a non-zero entry. If two neighboring columns are colored, no rotation symmetry exists. If we have a pattern with equally separated colored columns, the distance inbetween specifies the maximal u. But in general it is only of interest, which column has non-zero entries. Let us label the columns by the corresponding rotation speed and let S be the set of all differences in the labels to our colored columns. Then $\gcd(R) = \gcd(S)$.

If a $c_{kk} \neq 0$, then the smallest absolute rotation speed r of all non-vanishing coefficients c_{pq} with $p \neq q$ is an upper bound for the maximal u. If all c_{kk} vanish, then $2r$ is an upper bound. All this can be used to efficiently calculate whether a curve F is u-fold rotational symmetric or not.

With the above knowledge, we can determine rotation symmetry with respect to rotations around the origin. If we are interested in rotation symmetry with an arbitrary center, we may use the following facts (cf. [3]):

Lemma 3. *Let F be a a rotational symmetric curve with respect to the origin as center of rotation. Further let $c_{pq} \neq 0$ be a non-vanishing coefficient of F. Then $c_{p-1,q} = c_{p,q-1} = 0$ vanish, provided that $p-1, q-1 \geq 0$.*

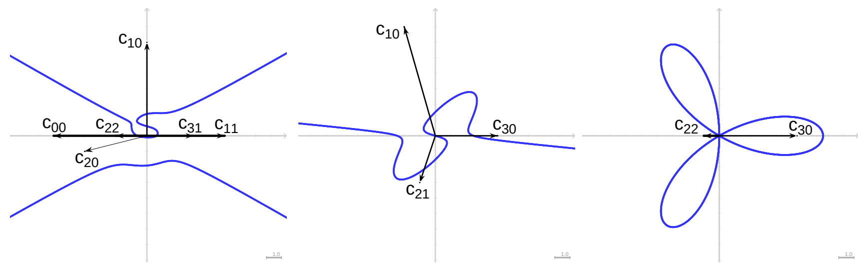
Proof. Let $c_{pq} \neq 0$ be a coefficient of g. Its rotation speed is $q - p$. If $c_{p-1,q}$ or $c_{p,q-1}$ would be non-zero then the corresponding rotation speed would differ by one from $q - p$. Consequently a u-fold rotational symmetry of F is bounded to $u = 1$ (see Theorem 8). Thus we can not speak of rotation symmetry. □

Theorem 9. *Let F be a curve, whose leading coefficients are not of the same absolute value. By Theorem 5 there is a translation such that the translated copy of F has a vanishing subleading coefficient. Let $\widetilde{F}$ be a copy of F, which had been translated by such a transformation. Then F is u-fold rotational symmetric with respect to an arbitrary center if and only if $\widetilde{F}$ is u-fold rotational symmetric with respect to the origin as center of rotation.*

Proof. If F is not rotational symmetric then any translated copy of F is also not rotational symmetric. But if F is in fact rotational symmetric, then there naturally exists a translation such that the center of the rotational symmetry is mapped to the origin. In that state the curve - let us name it F' - must satisfy the above lemmata. Now F is a curve, whose leading coefficients are not of the same absolute value. Let $c_{k+1,l}$ and $c_{k,l+1}$ be an arbitrary pair of succeeding leading coefficients with different absolute values. Then at least one of them is non-zero. Of course the same is true for one of the coefficients $c'_{k+1,l}$ and $c'_{k,l+1}$ of F'. (Translations do not change leading coefficients.) Thus by Lemma 3 the coefficient c'_{kl} must vanish. But by Theorem 5 there is exactly one translation annihilating c_{kl}. Consequently annihilation of an arbitrary subleading coefficient sends the center of rotational symmetry onto the origin, if F is rotational symmetric. □

This shows that our translatorial normal form moves a potentially existing but unknown center of rotational symmetry to the origin. The clue was that the translation annihilating a single coefficient was uniquely determined. For curves with leading coefficients of the same absolute value this is not true. There the notion of normal form has to be extended. One can do so by defining that such a curve is in canonical form, if *two* well chosen coefficients are of minimal absolute value. Then the corresponding translation will be unique again and a theorem corresponding to Theorem 9 can be formulated (see [3]).

Altogether we are able to detect rotation symmetry by looking at the curve parameter vector. Figure 1 and 4 show examples of rotational symmetric curves along with the corresponding complex parameters.



$c_{00}, c_{10} \neq 0$ $\Rightarrow$ no rotational symmetry — only $c_{10}, c_{21}, c_{30} \neq 0$ $\Rightarrow$ 2-fold symmetry — only $c_{22}, c_{30} \neq 0$ $\Rightarrow$ 3-fold symmetry

Fig. 4. Rotational symmetry and normal-form; All curves above are in rotational and translatorial normal form

5 Examples

Let us review the preceding sections by looking at two curves of degree four. Multinomial coefficients are denoted by $m_{kl} = \frac{4!}{k!\ l!\ (4-k-l)!}$.

- $\mathcal{C}_{f_1}$: $f_1(x,y) = \sum_{k+l\leq 4} A^{(1)}_{kl} x^k y^l = \sum_{k+l\leq 4} m_{kl}\ a^{(1)}_{kl}\ x^k y^l$ with the coefficient vector $a^{(1)} = (a^{(1)}_{00}, a^{(1)}_{10}, ..., a^{(1)}_{04})$ given by

$$(11906, -1958, -806, 350, 76, 154, -62, -14, -2, -50, 10, 4, -2, 4, 10)$$

- $\mathcal{C}_{f_2}$: $f_2(x,y) = \sum_{k+l\leq 4} A^{(2)}_{kl} x^k y^l = \sum_{k+l\leq 4} m_{kl}\ a^{(2)}_{kl}\ x^k y^l$ with the coefficient vector $a^{(2)} = (a^{(2)}_{00}, a^{(2)}_{10}, ..., a^{(2)}_{04})$ given by

$$(158, -168, -36, 60, 24, 20, -6, -12, -2, -12, 6, 0, 2, 0, 6)$$

These curves can be seen in Figure 5 together with transformed versions.

After complexification we get two complex parameter vectors $c^{(1)}$ for f_1 and $c^{(2)}$ for f_2. We notate them in for of coefficient triangles (see (13)):

$block_t(c^{(1)})$	curve coefficients								
$t=0$					11906				
$t=1$				-979 $+403i$		-979 $-403i$			
$t=2$			$49-38i$		126		$49+38i$		
$t=3$		$-7-i$		$-8+8i$		$-8-8i$		$-7+i$	
$t=4$	2		$-i$		1		i		2
rotation speed	-4	-3	-2	-1	0	1	2	3	4

$block_t(c^{(2)})$	curve coefficients								
$t=0$					158				
$t=1$				$-84+18i$		$-84-18i$			
$t=2$			$10-12i$		20		$10+12i$		
$t=3$		$3i$		$-1+3i$		$-1-3i$		$-3i$	
$t=4$	0		0		1		0		0
rotation speed	-4	-3	-2	-1	0	1	2	3	4

Calculating the translatorial normal form of for f_1 means annihilating the coefficient $c^{(1)}_{30}$ by a translation with $t_1 = -5-3i$. The translatorial normal form for f_2 is obtained by a translation of $t_2 = -1-3i$, annihilating $c^{(2)}_{21}$. We name the translated curve $\widetilde{f}_1$ and $\widetilde{f}_2$. The corresponding parameter vectors are given by

$block_t(\widetilde{c}^{(1)})$	curve coefficients								
$t=0$					**10**				
$t=1$				0		0			
$t=2$			**1**		**-2**		**1**		
$t=3$		0		0		0		0	
$t=4$	**2**		**-i**		**1**		**i**		**2**
rot. speed	**-4**	-3	**-2**	-1	**0**	1	**2**	3	**4**

and

$block_t(\widetilde{c}^{(2)})$	curve coefficients								
$t=0$					**-10**				
$t=1$				0		0			
$t=2$			0		0		0		
$t=3$		**3i**		0		0		**-3i**	
$t=4$	0		0		**1**		0		0
rot. speed	-4	**-3**	-2	-1	**0**	1	2	**3**	4

looking at the structure of the non-vanishing coefficients in the corresponding coefficient triangles, we can directly see the rotational symmetry: $\widetilde{f}_1$ and thus f_1 is rotational 2-symmetric and f_2 is rotational 3-symmetric.

The rotational normal form of $\widetilde{f}_1$ can be obtained in the following way: $\widehat{c}^{(1)}_{40}$ is already contained in $\mathbb{R}_{>0}$. Rotations by $\varphi \in \{0, \frac{\pi}{2}, \pi, \frac{3\pi}{4}\}$ preserve this property. But $\widetilde{c}^{(1)}_{31}$ may be changed. If $\varphi \in \{\frac{\pi}{2}, \frac{3\pi}{2}\}$, then the coefficient $\widehat{c}^{(1)}_{31}$ of the rotated copy $\widehat{f}_1$ of $\widetilde{f}_1$ changes its sign: $\widehat{c}^{(1)}_{31} = \widetilde{c}^{(1)}_{31} e^{-2\varphi i} = -\widetilde{c}^{(1)}_{31}$. If $\varphi \in \{0, \pi\}$, then $\widehat{c}^{(1)}_{31} = \widetilde{c}^{(1)}_{31}$. Consequently $\widehat{c}^{(1)}_{31} \in \{\pm i\}$. Writing $\widehat{c}^{(1)}_{31}$ in form of $re^{i\psi}$ with $\psi \in [0, 2\pi)$, we can see that $\psi \in \{\frac{\pi}{2}, \frac{3\pi}{2}\}$ and $\frac{\pi}{2} < \frac{3\pi}{2}$. This means that we have to rotate the curve by an angle φ, which implies $\psi = \frac{\pi}{2}$. Consequently $\varphi \in \{\frac{\pi}{2}, \frac{3\pi}{2}\}$. φ is not uniquely determined. So we continue with the next relevant non-vanishing coefficient: $c^{(2)}_{20}$. It has the same angular velocity as $c^{(1)}_{31}$ and therefore no new restriction on φ can be implied. Thus we end up with two possible rotations to transform $\widetilde{f}_1$ into rotational normal form. This confirms again the 2-fold rotational symmetry. Both $\varphi \in \{\frac{\pi}{2}, \frac{3\pi}{2}\}$ give us the same coefficient vector of $\widehat{f}_1$:

$$\widehat{c}^{(1)} = (10, 0, 0, -1, -2, -1, 0, 0, 0, 0, 2, i, 1, -i, 2) \quad .$$

Similar calculations for $\widetilde{f}_2$ lead to rotations by $\varphi \in \{\frac{\pi}{6}, \frac{5\pi}{6}, \frac{3\pi}{2}\}$, all leading to a positive and real $c^{(2)}_{30}$. The final parameter vector for $\widehat{f}_2$ is

$$\widehat{c}^{(2)} = (-10, 0, 0, 0, 0, 0, 3, 0, 0, 3, 0, 0, 1, 0, 0) \quad .$$

Thus finally $\widehat{f}_1$ and $\widehat{f}_2$ are in rotational and translational normal form and thus in normal form with respect to orientation preserving euclidean motions of the plane.

6 Conclusion

We have demonstrated how linear transformations of the plane affect curves contained in this plane. A special transformation had been the complex reparametrization of curves. We have seen that complexified curves are a lot easier to handle than curves with real coefficients: A rotation around the origin of a curve can be expressed by rotating the curve coefficients. In addition also translations are easy to describe. The curve coefficients transform by an upper triangular matrix with identity blocks in the diagonal. Subleading coefficients transform linearly in the translation parameter and are easily annihilated by translations. Making a subleading coefficient vanish means transforming a curve into normal form. It does not destroy the rotatory normal-form of having a positive real leading coefficient. Additionally rotation symmetry can be detected. Studying rotation symmetry with the origin as center is sufficient when the curve is in translatorial normal form.

Work on normal forms and invariants with respect to other transformation groups is in progress. Euclidean and similarity invariants and normal forms seem to be relatively simple to access but elliptic or projective ones not. There the approach of analyzing the transformation matrix for the curve coefficients is not at all practical. Another shift in the representation of curves by use of tensors and tensor diagrams seems promising.

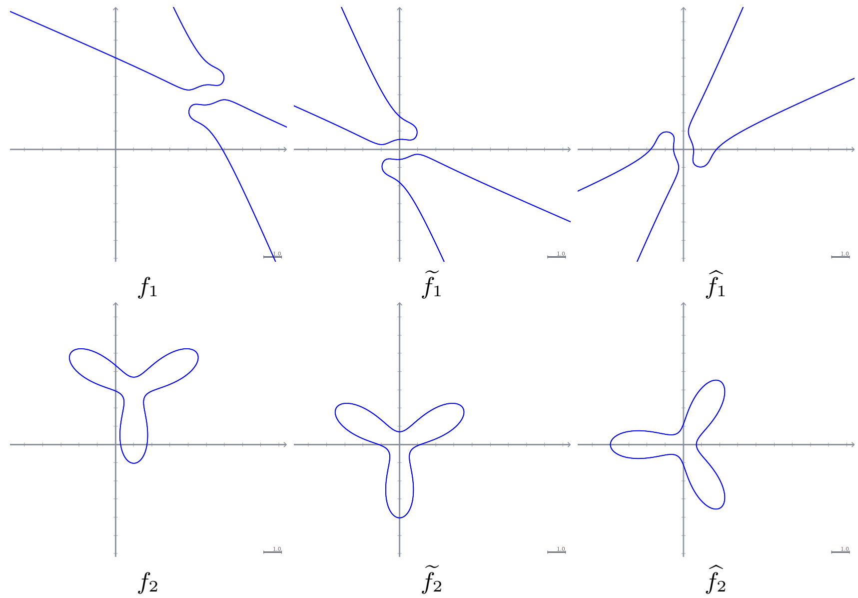


Fig. 5. Two curves under observation

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